A DEPENDENT THEORY WITH FEW INDISCERNIBLES

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Abstract. We continue the work started in [KS12], and prove the following theorem: for every θ there is a dependent theory T of size θ such that for all κ and δ, $\kappa \to (\delta)_{T,1}$ iff $\kappa \to (\delta)_{\theta}^{<\omega}$. This means that unless there are good set theoretical reasons, there are large sets with no indiscernible sequences.

1. Introduction

In the summer of 2008, Saharon Shelah announced in a talk in Rutgers that he had proved some very important results in dependent (NIP) theories. One of these was the existence of indiscernible, an old conjuncture of his. Here is the definition:

Definition 1.1. Let T be a theory. For a cardinal κ, $n \leq \omega$ and an ordinal δ, $\kappa \to (\delta)_{T,n}$ means: for every set $A \subseteq \mathfrak{C}^n$ of size κ, there is a non-constant sequence of elements of A of length δ which is indiscernible.

In stable theories, it is known that for any λ satisfying $\lambda = \lambda^{|T|}$, $\lambda^+ \to (\lambda^+)_{T,n}$ (proved by Shelah in [She90], and follows from local character of non-forking). The first claim of such nature was proved by Morley in [Mor65] for ω-stable theories. Later, in [She12], Shelah proved:

Fact 1.2. *If T is strongly dependent*[1]*, then for all $\lambda \geq |T|$, $\beth_{|T|^+}(\lambda) \to (\lambda^+)_{T,n}$ for all $n < \omega$.*

This definition was suggested by Grossberg and Shelah and appears in [She86, pg. 208, Definition 3.1(2)], in a slightly different form[2]. There it is also proved that such a theorem does not hold for simple unstable theories in general. So the natural generalization is to NIP theories, and indeed it is conjectured already in [She86, pg. 209, Conjecture 3.3].

This conjuncture is connected to a result by Shelah and Cohen: in [CS09], they proved that a theory is stable iff it can be presented in some sense in a free algebra in a fix vocabulary but allowing function symbols with infinite arity. If this result could be extended to: a theory is

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[1]For more on strongly dependent theories, see Section 6.

[2]The definition there is: $\kappa \to (\delta)_{n,T}$ if and only if for each sequence of length κ (of n-tuples), there is an indiscernible sub-sequence of length δ. For us there is no difference because we are dealing with examples where $\kappa \not\to (\mu)_{T,n}$. It is also not hard to see that when δ is a cardinal these two definitions are equivalent.

dependent iff it can be represented as an algebra with ordering, then this could be used to prove existence of indiscernibles.

Despite announcing it, there was a mistake in the proof for dependent theories, and here we shall see a counter-example. In the previous paper [KS12], we have shown that:

Theorem 1.3. *There exists a countable dependent theory T such that:*

For any two cardinals $\mu \leq \kappa$ with no (uncountable) strongly inaccessible cardinals in $[\mu, \kappa]$, $\kappa \nrightarrow (\mu)_{T,1}$.

And in this paper we improve it by:

Theorem 1.4. *For every θ there is a dependent theory T of size θ such that for all κ and δ, $\kappa \to (\delta)_{T,1}$ iff $\kappa \to (\delta)_{\theta}^{<\omega}$.*

Where:

Definition 1.5. $\kappa \to (\delta)_{\theta}^{<\omega}$ means: for every coloring $c : [\kappa]^{<\omega} \to \theta$ there is a homogeneous sub-sequence of length δ (i.e. there exists $\langle \alpha_i \,|\, i < \delta \rangle \in {}^{\delta}\kappa$ and $\langle c_n \,|\, n < \omega \rangle \in {}^{\omega}\theta$ such that $c\left(\alpha_{i_0}, \ldots, \alpha_{i_{n-1}}\right) = c_n$ for every $i_0 < \ldots < i_{n-1} < \delta$).

One can see that $\kappa \to (\delta)_{\theta}^{<\omega}$ implies $\kappa \to (\delta)_{T,1}$, so this is the best result possible.

We also should note that a related result can be found in an unpublished paper in Russian by Kudajbergenov that states that for every ordinal α there exists a dependent theory (but it may be even strongly dependent) T_α such that $|T_\alpha| = |\alpha| + \aleph_0$ and $\beth_\alpha(|T_\alpha|) \nrightarrow (\aleph_0)_{T_\alpha,1}$ and thus seem to indicate that the bound in Fact 1.2 is tight.

1.1. **The idea of the proof.** The example is a "tree of trees" with functions between the trees. More precisely, for all η in the base tree ${}^{\omega \geq}2$ we have a unary predicate P_η and an ordering $<_\eta$ such that $(P_\eta, <_\eta)$ is a discrete tree. In addition we will have functions $G_{\eta,\eta\hat{}\{i\}} : P_\eta \to P_{\eta\hat{}\langle i \rangle}$ for $i = 0, 1$. The idea is to prove that if $\kappa \nrightarrow (\delta)_{\theta}^{<\omega}$ then $\kappa \nrightarrow (\delta)_{T,1}$ by induction on κ, i.e. to prove that we can find a subset of $P_{\langle\rangle}$ of size κ without an indiscernible sequence in it. For κ regular but not strongly inaccessible or κ singular the proof is similar to the one in [KS12]: we just push our previous examples into deeper levels.

The main case is when κ is strongly inaccessible.

We have a coloring c that witnesses that $\kappa \nrightarrow (\delta)_{\theta}^{<\omega}$ and we build a model M_c that uses it. In this model, the base tree will be ω and not ${}^{\omega \geq}2$, i.e. for each $n < \omega$ we have a predicate P_n with tree-ordering $<_n$ and functions $G_n : P_n \to P_{n+1}$. In addition, $P_0 \subseteq \kappa$. On P_n we will define an equivalence relation E_n refining the neighboring relation (x, y are neighbors if they succeed the same element) so that every class of neighbors (neighborhood) is a disjoint union of less than κ many classes of E_n. We will prove that if there are indiscernibles in P_0, then there is some $n < \omega$

such that in P_n we get an indiscernible sequence $\langle t_i \mid i < \delta \rangle$ that looks like a fan, i.e. there is some u such that $t_i \wedge t_j = u$ and t_i is the successor of u, and in addition t_i and t_j are not E_n equivalent for $i \neq j$.

Now embed M_c in a model of our theory (i.e. now the base tree is again ${}^{\omega \geq}2$), and in each neighborhood we send every E_n class to an element from the model we get from the induction hypothesis (as there are less than κ many classes, it is possible).

By induction, we get there is no indiscernible sequence in P_0 and finish.

1.2. **Description of the paper.** In Section 2 we give some preliminaries on dependent theories and trees. In Section 3 we describe the theory and prove quantifier elimination and dependence. In Section 4 we prove the theorem up to inaccessible cardinals, and in Section 5 we finish the proof. In Section 6 we explain some of the choices we made during the constructions and discuss strongly dependent theories.

2. Preliminaries

Notation.

We use standard notation. a, b, c are elements, and $\bar{a}, \bar{b}, \bar{c}$ are finite or infinite tuples of elements.

$\mathfrak{C}$ will be the monster model of the theory.

$S_n(A)$ is the set of complete types over A, and $S_n^{\mathrm{qe}}(A)$ is the set of all quantifier free complete types over A. For a finite set of formulas with a partition of variables, $\Delta(\bar{x}; \bar{y})$, $S_{\Delta(\bar{x};\bar{y})}(A)$ is the set of all Δ-types over A, i.e. maximal consistent subsets of

$\{\varphi(\bar{x}, \bar{a}), \neg\varphi(\bar{x}, \bar{a}) \mid \varphi(\bar{x}, \bar{y}) \in \Delta \,\&\, \bar{a} \in A\}$[3]. Similarly we define $\mathrm{tp}_{\Delta(\bar{x};\bar{y})}\left(\bar{b}/A\right)$ as the set of formulas $\varphi(\bar{x}, \bar{a})$ such that $\varphi(\bar{x}, \bar{y}) \in \Delta$ and $\mathfrak{C} \models \varphi\left(\bar{b}, \bar{a}\right)$.

Dependent theories.

For completeness, we give here the definitions and basic facts we need on dependent theories.

Definition 2.1. A first order theory T is dependent if it does not have the independence property which means: there is no formula $\varphi(\bar{x}, \bar{y})$ and elements in $\left\langle \bar{a}_i, \bar{b}_s \mid i < \omega, s \subseteq \omega \right\rangle$ in $\mathfrak{C}$ such that $\varphi\left(\bar{a}_i, \bar{b}_s\right)$ iff $i \in s$.

We shall need the following fact:

Fact 2.2. [She90, II, 4] *Let T be any theory. Then for all $n < \omega$, T is dependent if and only if $\Box_n$ if and only if $\Box_1$ where for all $n < \omega$,*

- $\Box_n$ For every finite set of formulas $\Delta(\bar{x}, \bar{y})$ with $n = \lg(x)$, there is a polynomial f such that for every finite set $A \subseteq M \models T$, $|S_\Delta(A)| \leq f(|A|)$.

[3] $\bar{a} \in A$ means: $\bar{a}$ is a tuple of elements from A.

Trees.

Let us remind the reader of the basic definitions and properties of trees.

Definition 2.3. A tree is a partially ordered set $(A,<)$ such that for all $a \in A$, the set $A_{<a} = \{x \mid x < a\}$ is linearly ordered.

Definition 2.4. We say that a tree A is well ordered if $A_{<a}$ is well ordered for every $a \in A$. Assume now that A is well ordered.

- For every $a \in A$, denote $\operatorname{lev}(a) = \operatorname{otp}(A_{<a})$ — the level of a is the order type of $A_{<a}$.
- The height of A is $\sup\{\operatorname{lev}(a) \mid a \in A\}$.
- $a \in A$ is a root if it is minimal.
- A is normal when for all limit ordinals δ, and for all $a, b \in A$ such that 1) $\operatorname{lev}(a) = \operatorname{lev}(b) = \delta$, and 2) $A_{<a} = A_{<b}$, $a = b$.
- If $a < b$ then we denote by $\operatorname{suc}(a,b)$ the successor of a in the direction of b, i.e. $\min\{c \leq b \mid a < c\}$.
- We write $a <_{\operatorname{suc}} b$ if $b = \operatorname{suc}(a,b)$.
- We call A standard if it is well ordered, normal, and has a root.

For a standard tree $(A,<)$, define $a \wedge b = \max\{c \mid c \leq a \,\&\, c \leq b\}$.

3. Construction of the example

The first order theory.

The language:

Let S be a standard tree, and let L_S be the language:

$$\left\{P_\eta, <_\eta, \wedge_\eta, G_{\eta_1,\eta_2}, \operatorname{suc}_\eta, \operatorname{pre}_\eta, \lim_\eta \mid \eta, \eta_1, \eta_2 \in S, \eta_1 <_{\operatorname{suc}} \eta_2\right\}.$$

Where:

- P_η is a unary predicate, $<_\eta$ is a binary relation symbol, $\wedge_\eta$ and suc_η are 2 place function symbols, G_{η_1,η_2}, pre_η and $\lim_\eta$ are 1 place function symbols.

Definition 3.1. Let $L'_S = \{P_\eta, <_\eta, \wedge_\eta, G_{\eta_1,\eta_2}, \lim_\eta \mid \eta \in S, \eta_1 <_{\operatorname{suc}} \eta_2 \in S\}$ (i.e. L_S without pre and suc).

The theory:

Definition 3.2. The theory $T_S^\forall$ says:

- $(P_\eta, <_\eta)$ is a tree.
- $\eta_1 \neq \eta_2 \Rightarrow P_{\eta_1} \cap P_{\eta_2} = \emptyset$.
- $\wedge_\eta$ is the meet function: $x \wedge_\eta y = \max\{z \in P_\eta \mid z \leq_\eta x \,\&\, z \leq_\eta y\}$ for $x, y \in P_\eta$ (so its existence is part of the theory).

- suc_η is the successor function — for $x, y \in P_\eta$ with $x <_\eta y$, $\mathrm{suc}_\eta(x, y)$ is the successor of x in the direction of y (i.e. the trees are discrete). The axioms are:
  - $\forall x <_\eta y\, (x <_\eta \mathrm{suc}_\eta(x, y) \leq_\eta y)$, and
  - $\forall x \leq_\eta z \leq_\eta \mathrm{suc}_\eta(x, y)\, [z = x \vee z = \mathrm{suc}_\eta(x, y)]$.
- $\lim_\eta(x)$ is the greatest limit element below x. Formally,
  - $\lim_\eta : P_\eta \to P_\eta$, $\forall x \lim_\eta(x) \leq_\eta x$, $\forall x <_\eta y\, (\lim_\eta(x) \leq_\eta \lim_\eta(y))$,
  - $\forall x <_\eta y\, (\lim_\eta(\mathrm{suc}_\eta(x, y)) = \lim_\eta(x))$, $\forall x \lim_\eta(\lim_\eta(x)) = \lim_\eta(x)$.
- Define that x is a successor if $\lim_\eta(x) <_\eta x$, and denote

$$\mathrm{Suc}(P_\eta) = \{x \in P_\eta \mid \lim_\eta(x) <_\eta x\}.$$

- pre_η is the immediate predecessor function —
  $\forall x \neq \lim_\eta(x)\, \left(\mathrm{pre}_\eta(x) < x \wedge \mathrm{suc}_\eta\left(\mathrm{pre}_\eta(x), x\right) = x\right)$.
- (**regressive**) If $\eta_1 <_{\mathrm{suc}} \eta_2$ then G_{η_1,η_2} satisfies: $G_{\eta_1,\eta_2} : \mathrm{Suc}(P_{\eta_1}) \to P_{\eta_2}$ and if $x <_{\eta_1} y$, both x and y are successors, and $\lim_\eta(x) = \lim_\eta(y)$ then $G_{\eta_1,\eta_2}(x) = G_{\eta_1,\eta_2}(y)$.

Example 3.3. Assume that $(M, <)$ is a standard tree. Then M has a natural structure for the language $L_{\{\emptyset\}}$ (where $\lim_\emptyset^M(a)$ is the greatest element below a of a limit level).

Model completion.

Here we prove that this theory has a model completion. As models may be infinite, we cannot use Fraïssé's theorem.

If S_1, S_2 are standard trees, we shall treat them as structures in the language $\{<_{\mathrm{suc}}, <\}$, so when we write $S_1 \subseteq S_2$, we mean that S_1 is a substructure of S_2 in this language. Sometime we write $<$, suc instead of suc_η, $<_\eta$ or suc_η^M, $<_\eta^M$ where M and η are clear from the context.

Remark 3.4.

(1) $T_S^\forall$ is a universal theory.
(2) $T_S^\forall$ has the joint embedding property (JEP).
(3) If $S_1 \subseteq S_2$ then $T_{S_1}^\forall \subseteq T_{S_2}^\forall$ and moreover, if $M \models T_{S_2}^\forall$ is existentially closed, $M \restriction L_{S_1}$ is an existentially closed model of $T_{S_1}^\forall$.

Assume from now until Corollary 3.25 below that S is finite.

Definition 3.5.

(1) Suppose Σ is a finite set of terms from L_S. We define the following closure operators on terms:
  (a) $\mathrm{cl}_\wedge(\Sigma) = \Sigma \cup \bigcup\left\{\wedge_\eta\left(\Sigma^2\right) \mid \eta \in S\right\} = \Sigma \cup \{t_1 \wedge_\eta t_2 \mid t_1, t_2 \in \Sigma\}$.
  (b) $\mathrm{cl}_G(\Sigma) = \Sigma \cup \bigcup\{G_{\eta_1,\eta_2}(\Sigma) \mid \eta_1 <_{\mathrm{suc}} \eta_2 \in S\}$.
  (c) $\mathrm{cl}_{\lim}(\Sigma) = \Sigma \cup \bigcup\{\lim_\eta(\Sigma) \mid \eta \in S\}$.

(d) $\mathrm{cl}^0(\Sigma) = \mathrm{cl}_G(\mathrm{cl}_{\lim}(\mathrm{cl}_\wedge(\ldots(\mathrm{cl}_G(\mathrm{cl}_{\lim}(\mathrm{cl}_\wedge(\Sigma)))))))$ where the number of compositions is the length of the longest branch in S.

(e) $\mathrm{cl}_{\mathrm{suc}}(\Sigma) = \bigcup\left\{\mathrm{suc}_\eta\left(\Sigma^2\right) \cup \mathrm{pre}_\eta(\Sigma) \,|\, \eta \in S\right\} \cup \Sigma$.

(f) $\mathrm{cl}(\Sigma) = \mathrm{cl}^0(\mathrm{cl}_{\mathrm{suc}}(\Sigma))$.

(2) Denote $\mathrm{cl}^{(0)} = \mathrm{cl}^0$ and for a number $k < \omega$, $\mathrm{cl}^{(k)}(A) = \mathrm{cl}(\ldots\mathrm{cl}(\Sigma))$ where the composition is done k times.

In the same way, if $\bar{t} = \langle t_i \,|\, i < n\rangle$ is an n-tuple of terms then $\mathrm{cl}(\bar{t})$ is $\mathrm{cl}(\{t_i \,|\, i < n\})$. If $S' \subseteq S$, we define $\mathrm{cl}_{S'}(\Sigma)$ by restricting all operations to those from $L_{S'}$.

(3) For a model $M \models T_S^\forall$, and $\bar{a} \in M$, define $\mathrm{cl}^0(\bar{a}) = \left(\mathrm{cl}^0(\bar{x})\right)^M(\bar{a})$ where $\bar{x}$ is a sequence of variables in the length of $\bar{a}$. In the same way define $\mathrm{cl}_\wedge(\bar{a}), \mathrm{cl}_{\lim}(\bar{a}), \mathrm{cl}_G(\bar{a}), \mathrm{cl}_{\mathrm{suc}}(\bar{a})$ and $\mathrm{cl}^{(k)}(\bar{a})$. For a set $A \subseteq M$, define $\mathrm{cl}^0(A) = \mathrm{cl}^0(\bar{a})$ where $\bar{a}$ is an enumeration of A, and in the same way the other closure operators.

Claim 3.6. For $A \subseteq M$, $\mathrm{cl}^0(A)$ is closed under $\wedge_\eta$, $\lim_\eta$ and G_{η_1,η_2}. So it is the substructure generated by A in the language L'_S.

Proof. It is east to see that $\mathrm{cl}_{\lim}(\mathrm{cl}_\wedge(A))$ is closed under $\lim_\eta$ and $\wedge_\eta$ for all $\eta \in S$.

For $n < \omega$, let $\mathrm{cl}^0_n(A) = \mathrm{cl}_G(\mathrm{cl}_{\lim}(\mathrm{cl}_\wedge(\ldots(\mathrm{cl}_G(\mathrm{cl}_{\lim}(\mathrm{cl}_\wedge(A)))))))$ where there are n compositions. For $\eta \in S$, let $r(\eta) = r_S(\eta) = |\{\nu \in S \,|\, \nu \leq \eta\}|$, so $\mathrm{cl}^0 = \mathrm{cl}^0_{\max\{r(\eta)|\,\eta\in S\}}$. Let $B \supseteq A$ be the closure of A under these operations in M. Then B is in fact $\mathrm{cl}^0_\omega(A) = \bigcup\left\{\mathrm{cl}^0_n(A) \,|\, n < \omega\right\}$. One can prove by induction on $r(\eta)$ that $B \cap P_\eta = \mathrm{cl}^0_{r(\eta)}(A) \cap P_\eta$. Hence $B = \mathrm{cl}^0(A)$. □

Claim 3.7. For every $k < \omega$, there is a polynomial f_k such that for every finite A, $\left|\mathrm{cl}^{(k)}(A)\right| \leq f_k(|A|)$.

Proof. Obvious. □

Definition 3.8.

(1) For a term t, we define its *successor rank* as follows: if suc and pre do not appear in t, then $r_{\mathrm{suc}}(t) = 0$. For two terms t_1, t_2: $r_{\mathrm{suc}}(\mathrm{suc}_\eta(t_1, t_2)) = \max\{r_{\mathrm{suc}}(t_1), r_{\mathrm{suc}}(t_2)\} + 1$, $r_{\mathrm{suc}}\left(\mathrm{pre}_\eta(t_1)\right) = r_{\mathrm{suc}}(t_1) + 1$, $r_{\mathrm{suc}}(t_1 \wedge t_2) = \max\{r_{\mathrm{suc}}(t_1), r_{\mathrm{suc}}(t_2)\}$, $r_{\mathrm{suc}}(G_{\eta_1,\eta_2}(t_1)) = r_{\mathrm{suc}}(t_1)$ and $r_{\mathrm{suc}}(\lim_\eta(t_1)) = r_{\mathrm{suc}}(t_1)$.

(2) For a quantifier free formula φ, let $r_{\mathrm{suc}}(\varphi)$ be the maximal rank of a term appearing in φ.

(3) For $k < \omega$ and an n-tuple of variables $\bar{x}$, denote by $\Delta^{\bar{x}}_k$ the set of all atomic formulas $\varphi(\bar{x})$ such that for every term t in φ, $t \in \mathrm{cl}^{(k)}(\bar{x})$. Note that since $\mathrm{cl}^{(k)}(\bar{x})$ is a finite set, so is $\Delta^{\bar{x}}_k$.

Claim 3.9. Assume that $M \models T_S^\forall$ and $\bar{a} \in M$. Then $\mathrm{cl}^{(k)}(\bar{a}) = \left\{t^M(\bar{a}) \,|\, r_{\mathrm{suc}}(t) \leq k\right\}$.

Proof. Easy (by induction on k). □

Definition 3.10. Assume that $M_1, M_2 \models T_S^{\forall}$ and $\bar{a} \in M_1$, $\bar{b} \in M_2$. For $k < \omega$, we say that $\bar{a} \equiv_k \bar{b}$ if there is an isomorphism of L'_S from $\mathrm{cl}^{(k)}(\bar{a})$ to $\mathrm{cl}^{(k)}(\bar{b})$ taking $\bar{a}$ to $\bar{b}$. For $S' \subseteq S$, denote $\bar{a} \equiv_k^{S'} \bar{b}$ when the isomorphism is of $L'_{S'}$ and from $\mathrm{cl}_{S'}^{(k)}(\bar{a})$ to $\mathrm{cl}_{S'}^{(k)}(\bar{b})$.

In this notation we assume that M_1, M_2 are clear from the context.

Definition 3.11. For $\bar{a} \in M \models T_S^{\forall}$ a finite tuple, $A \subseteq M$ a finite set, and $k < \omega$, let $\mathrm{tp}_k(\bar{a}/A) = \mathrm{tp}_{\Delta_k^{\bar{x}\bar{y}}(\bar{x};\bar{y})}(\bar{a}/A)$ where $\bar{y}$ is of length $|A|$. For $S' \subseteq S$, $\mathrm{tp}_k^{S'}(\bar{a}/A)$ is defined in the same way where we reduce to $L_{S'}$ formulas.

Claim 3.12. Assume $M_1, M_2 \models T_S^{\forall}$. Assume $\bar{a} \in M_1, \bar{b} \in M_2$ and $k < \omega$. Then $\bar{a} \equiv_k \bar{b}$ iff $\mathrm{tp}_k(\bar{a}) = \mathrm{tp}_k(\bar{b})$ iff for every quantifier free formula $\varphi(\bar{x})$ with $r_{\mathrm{suc}}(\varphi) \leq k$,

$$M_1 \models \varphi(\bar{a}) \Leftrightarrow M_2 \models \varphi(\bar{b}).$$

Proof. Assume $\bar{a} \equiv_k \bar{b}$ and $f : \mathrm{cl}^{(k)}(\bar{a}) \to \mathrm{cl}^{(k)}(\bar{b})$ is an L'_S-isomorphism taking $\bar{a}$ to $\bar{b}$. It is easy to see that for every term $t \in \mathrm{cl}^{(k)}(\bar{x})$, $f(t(\bar{a})) = t(\bar{b})$, and so $\mathrm{tp}_k(\bar{a}) = \mathrm{tp}_k(\bar{b})$.

On the other hand, if $\mathrm{tp}_k(\bar{a}) = \mathrm{tp}_k(\bar{b})$, then define f by: for every $t \in \mathrm{cl}^{(k)}(\bar{x})$, $f(t(\bar{a})) = t(\bar{b})$. As the types are equal, f is well defined and is an isomorphism.

The last statement is obviously strongly than the second one. The converse follows by Claim 3.9: for every term $t(\bar{x})$ with rank $r_{\mathrm{suc}}(t) \leq k$ there is a term $t' \in \mathrm{cl}^{(k)}(\bar{x})$ such that $M_1 \models t'(\bar{a}) = t(\bar{a})$. By induction on k and t, one can show that $M_2 \models t'(\bar{b}) = t(\bar{b})$ and that suffices. $\square$

Similarly, we have:

Claim 3.13.

(1) if $\bar{a} \equiv_k \bar{b}$ then there is a unique isomorphism that shows it.
(2) Assume $k_2 \geq k_1$. Then $\bar{a} \equiv_{k_2} \bar{b}$ implies $\bar{a} \equiv_{k_1} \bar{b}$.
(3) If $\bar{a}\bar{a}' \equiv_k \bar{b}\bar{b}'$ then $\bar{a} \equiv_k \bar{b}$.
(4) If $\bar{a} \equiv_{k+1} \bar{b}$ then $\mathrm{cl}(\bar{a}) \equiv_k \mathrm{cl}(\bar{b})$.
(5) Claim 3.12 and (1) − (4) are still true when we replace $\equiv_k$ by $\equiv_k^{S'}$ for $S' \subseteq S$ (with the obvious adjustments).

Definition 3.14. For $x <_\eta y$ from P_η in a model of $T_S^{\forall}$, we say that the distance between x and y is n if y is the n-th successor of x or vice-versa. We say the distance is infinite if for no $n < \omega$ the distance is n. Denote $d(x, y) = n$.

Lemma 3.15. *(Quantifier elimination lemma) For all $m_1, k < \omega$ there is $m_2 = m_2(m_1, k, S) < \omega$ such that if:*

- *$M_1, M_2 \models T_S^{\forall}$ are existentially closed.*
- *$\bar{a} \in M_1$ and $\bar{b} \in M_2$.*

- $\bar{a} \equiv_{m_2} \bar{b}$.

Then for all $\bar{c} \in {}^k M_1$ *there is some* $\bar{d} \in {}^k M_2$ *such that* $\bar{c}\bar{a} \equiv_{m_1} \bar{d}\bar{b}$.

Proof. Given S, it is enough to prove the lemma for $k = 1$, since we can define by induction $m_2(m_1, k+1, S) = m_2(m_2(m_1, k, S), 1, S)$.

The proof is by induction on $|S|$. We may assume that $m_2(m_1, k, S') > \max\{m_1, k, |S'|\}$ for $|S'| < |S|$ (by enlarging m_2 if necessary).

For $|S| = 0$ the claim is trivial because $T_S^{\forall}$ is just a theory of a set with no structure.

Assume $0 < |S|$. Let η_0 be the root of S and denote $S_0 = \{\eta_0\}$ and $S = \{\eta_0\} \cup \bigcup \{S_i \mid 1 \leq i < k\}$ where the S_i's are the connected components of S above η_0. Let

$$m_2 = m_2(m_1, 1, S) = \max\{2m_2(m_1, K, S_i) \mid 1 \leq i < k\} + 2m_1 + 1$$

where K is a natural number independent of m_1, k and S that will be described below.

Let $P_{S_i} = \bigvee (P_\eta \mid \eta \in S_i)$. Suppose M_1, M_2, $\bar{a}$ and $\bar{b}$ are as in the lemma.

Let $f : \mathrm{cl}^{(m_2)}(\bar{a}) \to \mathrm{cl}^{(m_2)}(\bar{b})$ be the isomorphism showing that $\bar{a} \equiv_{m_2} \bar{b}$ and we are given c.

For $1 \leq i \leq k$, let $A_i = \mathrm{cl}^{(m_1)}(\bar{a}) \cap P_{S_i}^{M_1}$ and $B_i = \mathrm{cl}^{(m_1)}(\bar{b}) \cap P_{S_i}^{M_1}$.

Since $\bar{a} \equiv_{m_2} \bar{b}$, it follows that $\mathrm{cl}^{(m_2(m_1,K,S_i))}(\bar{a}) \equiv_{m_2(m_1,K,S_i)} \mathrm{cl}^{(m_2(m_1,K,S_i))}(\bar{b})$ and in particular $A_i \equiv^{S_i}_{m_2(m_1,K,S_i)} B_i$ when we think of A_i and B_i enumerated in a way that f witnesses this.

We divide into cases: □

Case 1. $c \notin P_\eta^{M_1}$ for every $\eta \in S$.

Here finding d is easy.

Case 2. $c \in P_{S_i}^{M_1}$ for some $1 \leq i \leq k$.

$A_i \equiv^{S_i}_{m_2(m_1,K,S_i)} B_i$, so by the induction hypothesis (and by Remark 3.4 (3)) we can find $d \in M_2$ and expand $f \restriction \mathrm{cl}^{(m_1)}(A_i)$ to $f' : \mathrm{cl}^{(m_1)}(cA_i) \to \mathrm{cl}^{(m_1)}(dB_i)$. It follows that

$$f \restriction \mathrm{cl}^{(m_1)}(\bar{a}) \cup f' \restriction \mathrm{cl}^{(m_1)}(c\bar{a})$$

is an isomorphism from $\mathrm{cl}^{(m_1)}(c\bar{a})$ to $\mathrm{cl}^{(m_1)}(d\bar{b})$ that shows that $c\bar{a} \equiv_{m_1} d\bar{b}$ (note that $P_{S_j}^{M_1} \cap \mathrm{cl}^{(m_1)}(\bar{a}c) = A_j$ for $j \neq i$ and that if $x \in \mathrm{cl}^{(m_1)}(\bar{a}c) \cap P_{S_i}^{M_1}$ then $x \in \mathrm{cl}^{(m_1)}(cA_i)$, and so the domain is indeed $\mathrm{cl}^{(m_1)}(c\bar{a})$).

Case 3. $c \in P_{\eta_0}$.

Let $A_0 = \mathrm{cl}^{(0)}(\bar{a}) \cap P_{\eta_0}^{M_1}$, $B_0 = \mathrm{cl}^{(0)}(\bar{b}) \cap P_{\eta_0}^{M_2}$ and $\eta_i = \min(S_i)$ for $1 \leq i \leq k$. Note that $\mathrm{cl}(A_0) \cap P_{\eta_0}^{M_1} = \mathrm{cl}_{\mathrm{suc}}(A_0)$ and the same is true for $\mathrm{cl}^{(0)}(A_0 c)$. Let $F = \mathrm{cl}^{(0)}(A_0 c) \cap P_{\eta_0}^{M_1}$. For notational simplicity, let $<$ be $<_{\eta_0}$, lim be $\lim_{\eta_0}$ and so on.

The main idea is:

Say that an element of $\mathrm{cl}^{(m_1)}(A_0c) \cap P_{\eta_0} = \mathrm{cl}^{(m_1)}_{\mathrm{suc}}(F)$ is *new* if it is a successor and not in $\mathrm{cl}^{(m_1)}(A_0)$. If e is a new element, and it is in $\mathrm{cl}^{(r)}_{\mathrm{suc}}(F)$ for $1 < r$, then as for $1 \leq i \leq k$, the function G_{η_0,η_i} is regressive, there is some $e' \in \mathrm{cl}_{\mathrm{suc}}(F)$ such that $G_{\eta_0,\eta_i}(e) = G_{\eta_0,\eta_i}(e')$. By the case study below, there are at most K new elements in $\mathrm{cl}_{\mathrm{suc}}(F)$. Enumerate them by $\langle c_l \mid l < K \rangle$ and denote their images by $c^i_l = G_{\eta_0,\eta_i}(c_l)$. Fix $1 \leq i \leq k$. By assumption, $A_i \equiv^{S_i}_{m_2(m_1,N,S_i)} B_i$ (and f witnesses this), so by the induction hypothesis there is $d^i_l \in M_2$ for $l < K$ such that $\langle c^i_l \mid l < K \rangle A_i \equiv^{S_i}_{m_1} \langle d^i_l \mid l < K \rangle B_i$ as witnessed by some isomorphism g_i extending $f \upharpoonright A_i$. By choice of m_2 it follows (see below, in the division to subcases) that we can find a model of $T^{\forall}_{\{\eta_0\}}$, $M'_3 \supseteq P^{M_2}_{\eta_0}$ and some $d' \in M'_3$ such that $f \upharpoonright A_0$ can be extended to some f' showing that $A_0c \equiv^{S_0}_{m_1} B_0d'$ (we may assume that M'_3 is the minimal structure generated by $P^{M_2}_{\eta_0}$ and d'). Let M_3 be a model of $T^{\forall}_S$ satisfying $P^{M_3}_{\eta_0} = P^{M'_3}_{\eta_0}$, $M_3 \supseteq M_2$ and $G_{\eta_0,\eta_i}(f'(c_l)) = d^i_l$ (for $l < k$ and $1 \leq i \leq k$). It exists — all we need to check is that G_{η_0,η_i} is well defined, but this is clear because $A_0c \equiv^{S_0}_{m_1} B_0d'$ and $\langle c^i_l \mid l < K \rangle A_i \equiv^{S_i}_{m_1} \langle d^i_l \mid l < K \rangle B_i$.

Now it is easy to see that

$$f \upharpoonright \mathrm{cl}^{(m_1)}(\bar{a}) \cup f' \cup \bigcup \left\{ g_i \upharpoonright \mathrm{cl}^{(m_1)}(\bar{a}c) \mid 1 \leq i < k \right\}$$

is the isomorphism extending $f \upharpoonright \mathrm{cl}^{(m_1)}(\bar{a})$. So $c\bar{a} \equiv_{m_1} d'\bar{b}$, i.e. $\mathrm{tp}_{m_1}(c\bar{a}) = \mathrm{tp}_{m_1}(d'\bar{b})$, and if Ψ is the conjunction of all formulas appearing in $\mathrm{tp}_{m_1}(c\bar{a})$ then $M_3 \models \exists x \Psi(x\bar{b})$. As M_2 is existentially closed there is some $d \in M_2$ such that $\Psi(d\bar{b})$, i.e. $c\bar{a} \equiv_{m_1} d\bar{b}$.

Now we shall analyze the different subcases and show that there really is a bound K for the number of new elements in $\mathrm{cl}_{\mathrm{suc}}(Ac)$ and that we can find d' as above.

Case i. $c \in A_0$: obvious — there are no new elements.

Case ii. c is in a branch of A_0, i.e. there is $c < y \in A_0$ and assume y is minimal in this sense. We again divide into cases:

Case a. There is no $x \in A_0$ below c. The new elements from $\mathrm{cl}_{\mathrm{suc}}(F)$ are contained in $D = \{c, \mathrm{pre}(c), \mathrm{suc}(c,y), \mathrm{suc}(\lim(c), c)\}$. Now it is easy to see that d' and M'_3 exist (note that here $y = \lim(y)$).

Case b. There is some $x \in A_0$ such that $x < c$. Assume x is maximal in this sense. New elements again come from D.

If $\lim(x) < \lim(y)$ then necessarily $\lim(x) \leq x < c < \lim(y) = y$ and finding d' and M'_3 is easy.

If, on the other hand, $\lim(x) = \lim(y)$, we must make sure that the distance between $f(x)$ and $f(y)$ is big enough, so that we can place d' in the right spot between them. In $\mathrm{cl}^{(m_1)}_{\mathrm{suc}}(F) \setminus \mathrm{cl}^{(m_1)}_{\mathrm{suc}}(A_0)$

we may add m_1 successors to c in the direction of y and m_1 predecessors. This is why we chose $m_2 \geq 2m_1 + 1$.

Case iii. c starts a new branch in A_0, i.e. there is no $y \in A_0$ such that $c < y$. In this case, let $c' = \{\max(c \wedge b) \mid b \in A_0\}$. Note that if there is an element in $\mathrm{cl}_\wedge(A_0 c) \setminus A_0 c$, it must be c'. Adding c' falls under Subcase *ii* (if it is indeed new), so the new elements will be those which come from c' as before, and perhaps

$$\{c, \mathrm{pre}(c), \mathrm{suc}(c', c), \mathrm{suc}(\lim(c), c)\}.$$

By the previous case, we can find $M_3'' \supseteq P_{\eta_0}^{M_2}$ and $d'' \in M_3''$ such that $c' A_0 \equiv_{m_1} d'' B_0$, and then we can find $M_3' \supseteq M_3''$ and $d' \in M_3'$ such that $cA_0 \equiv_{m_1} d' B_0$ and we are done.

Claim 3.16. For every formula $\varphi(\bar{x})$ (with free variables) there is a quantifier free formula $\psi(\bar{x})$ such that for every existentially closed model $M \models T_S^\forall$, we have $M \models \psi \equiv \varphi$.

Proof. It is enough to check formulas of the form $\exists y \varphi(y, \bar{x})$ where φ is quantifier free and $\lg(\bar{x}) = n > 0$. Let $k = r_{\mathrm{suc}}(\varphi)$. Let $m = m_2(k, 1, S)$ from Lemma 3.15. By Claim 3.12, if $M_1, M_2 \models T_S^\forall$ are existentially closed and $\bar{a} \in M_1, \bar{b} \in M_2$ are of length n and $\bar{a} \equiv_m \bar{b}$, then $M_1 \models \exists y \varphi(y, \bar{a})$ iff $M_2 \models \exists y \varphi(y, \bar{b})$.

Assume $|\Delta_m^{\bar{x}}| = N$ and let $\{\varphi_i \mid i < N\}$ be an enumeration of $\Delta_m^{\bar{x}}$. For every $\eta : N \to 2$, let $\varphi_\eta^m(\bar{x}) = \bigwedge_{i<N} \varphi_i^{\eta(i)}(\bar{x})$ (where $\varphi^0 = \neg\varphi$ and $\varphi^1 = \varphi$).

Let

$$R = \left\{\eta : N \to 2 \,\middle|\, \exists \text{ e.c. } M \models T_S^\forall \,\&\, \exists \bar{c} \in M \left(M \models \varphi_\eta^m(\bar{c}) \wedge \exists y \varphi(y, \bar{c})\right)\right\}.$$

Let $\psi(\bar{x}) = \bigvee_{\eta \in R} \varphi_\eta^m(\bar{x})$. By Claim 3.12 it follows that ψ is the desired formula. $\square$

Corollary 3.17. *If M_1 and M_2 are two existentially closed models of $T_S^\forall$ then $M_1 \equiv M_2$ and their theory eliminates quantifiers.*

Proof. Assume first that $M_1 \subseteq M_2$, then $M_1 \prec M_2$: for formulas with free variables it follows directly from the previous claim, and for a sentence φ we consider the formula $\varphi \wedge (x = x)$.

Now the corollary follows from the fact that the theory is universal (so every model can be extended to an existentially closed one) and has JEP. $\square$

Definition 3.18. Let T_S be the theory of all existentially closed models of $T_S^\forall$. This is the model completion of $T_S^\forall$.

Dependence.

Now let us prove that T_S is dependent.

Definition 3.19. Assume $A \subseteq M \models T_S$ is a finite set and $k < \omega$.

(1) We say that $a, b \in M$ are k-isomorphic over A, denoted by $a \equiv_{A,k} b$ iff $aA \equiv_k bA$ (i.e. there is an isomorphism $f : \mathrm{cl}^{(k)}(aA) \to \mathrm{cl}^{(k)}(bA)$ of L'_S taking a to b and fixing A point-wise).
(2) Similarly for tuples from $M^{<\omega}$.
(3) Similarly for $S' \subseteq S$, and we shall write $a \equiv_{A,k}^{S'} b$ for $aA \equiv_k^{S'} bA$.

Claim 3.20. Assume $M \models T_S^{\forall}$, $k < \omega$, $A \subseteq M$ is finite and $\bar{a}, \bar{b} \in M$. Then $\bar{a} \equiv_{A,k} \bar{b}$ iff $\mathrm{tp}_k(\bar{a}/A) = \mathrm{tp}_k(\bar{b}/A)$ iff for every quantifier free formula over A, $\varphi(\bar{x})$ such that $r_{\mathrm{suc}}(\varphi) \leq k$, $M \models \varphi(\bar{a}) \leftrightarrow \varphi(\bar{b})$. The analogous claim is true for every $S' \subseteq S$ when we reduce to $L_{S'}$ formulas.

Proof. Follows from the definitions and from Claim 3.12. □

Claim 3.21. Assume $|S| = 1$ and $k < \omega$. Then there is a polynomial f such that for every finite set $A \subseteq M$, $|\{M/\equiv_{A,k}\}| \leq f(|A|)$.

Proof. As $|S| = 1$, we can forget the index η and write $<, \lim$, etc. instead of $<_\eta, \lim_\eta$, etc.

Given $a < b \in M$, the k-distance between them is defined by

$d_k(a, b) = \min\{d(a, b), 2k + 1\}$.

Assume $a \in M$ and $A \subseteq M$ is finite.

Let $B = \mathrm{cl}^0(A)$, and enumerate B by $\{b_i \mid i < l\}$. Recall that $l \leq f_0(|A|)$ (see Claim 3.7), hence it is polynomially bounded.

The following data is enough to determine the k-isomorphism type of a over A.

(1) Is $a \in P$ or not?
    (a) If not, then it is enough to know whether $a \in B$, and if yes, for which $i < l$ is it that $a = b_i$.
    If yes:
(2) What is the k-distance between a and $\lim(a)$?
(3) Is there some $i < l$ such that $a \leq b_i$?
    If yes:
    (a) Is $a = b_i$? if not, what is the minimal element such that this happens (recall that B is closed under $\wedge$, so if there is some $i < l$ such that $a < b_i$, there must be a minimal one). Call it b_i.
    (b) Is $\lim(a) = \lim(b_i)$?
        If yes:
        (i) What is the k-distance between a and b_i?
        (ii) What is the maximal b_j such that $b_j \leq a$?

(iii) What is the k-distance between a and b_j?

If not (i.e. $\lim(a) \neq \lim(b_i)$):

(i) Is there some b_j such that $b_j \leq a$?

If yes:

(A) What is the maximal such b_j?

(B) Is it true that $\lim(a) = \lim(b_j)$? If yes what is the k-distance between a and b_j?

If not (there is no b_i):

(a) Let $a' = \max\{a \wedge b_i \mid i < l\}$.

(b) Give the same information for a' (of course a' is smaller than or equal to some b_i). There is a polynomial bound on the number of k-isomorphism types for a' (by the analysis above).

(c) Is it true that $\lim(a) = \lim(a')$?

(i) If yes, what is the k-distance between a and a'?

That is enough. Since the number of possible answers for every question is polynomially bounded (in k and l), there is such a polynomial. $\square$

Claim 3.22. For every S the following are equivalent:

(1) T_S is dependent.

(2) For $k < \omega$ there is a polynomial f such that for every model $M \models T$ and finite $A \subseteq M$, $|\{M/\equiv_{A,k}\}| \leq f(|A|)$.

(3) For all $k, n < \omega$, there is a polynomial f such that for every model $M \models T$ and for every finite $A \subseteq M$, $|\{M^n/\equiv_{A,k}\}| \leq f(|A|)$.

Proof. The claim follows from Fact 2.2:

Let $A \subseteq M \models T_S$ be finite. Then $\bar{a} \equiv_{A,k} \bar{b}$ iff $\mathrm{tp}_k(\bar{a}/A) = \mathrm{tp}_k(\bar{b}/A)$ (by Claim 3.20). So (2) is equivalent to the claim that there is a polynomial bound on the size of $S_{\Delta_k^{\bar{x}\bar{y}}(\bar{x};\bar{y})}(A)$ (where $\lg(\bar{y}) = |A|$) in terms of $|A|$ when $\lg(\bar{x}) = 1$, and (3) for $\lg(\bar{x}) = n$. Hence (1) obviously implies (2) and (3).

Let Δ be a finite set of formulas in $\bar{x}\bar{y}$. By quantifier elimination, we may assume that Δ is quantifier free. Let $k = \max\{r_{\mathrm{suc}}(\varphi) \mid \varphi \in \Delta\}$ and $m = \left|S_{\Delta(\bar{x};\bar{y})}(A)\right|$. Let $\{\bar{c}_i \mid i < m\}$ be a set of tuples satisfying the all the different types in $S_{\Delta(\bar{x};\bar{y})}(A)$ in some model of T_S. If $i \neq j$ then $\mathrm{tp}_k(\bar{c}_i/A) \neq \mathrm{tp}_k(\bar{c}_j/A)$ (by Claim 3.20), so $m \leq \left|\left\{M^{\lg(\bar{x})}/\equiv_{A,k}\right\}\right|$, and hence (2) and (3) imply (1) by Fact 2.2. $\square$

Corollary 3.23. *T_S is dependent.*

Proof. The proof is by induction on $|S|$. For $|S| = 1$ it follows from Claim 3.21 and Claim 3.22 (and for $|S| = 0$ it is obvious).

Assume $1 < |S|$, and we shall show (2) from Claim 3.21. Assume we are given $k < \omega$.

Let η_0 be the root of S and let $S = \{\eta_0\} \cup \bigcup \{S_i \mid 1 \le i < m\}$ where the S_i's are the connected components of S above η_0. Denote $S_0 = \{\eta_0\}$, and $P_{S_i} = \bigvee (P_\eta \mid \eta \in S_i)$. Let $\eta_i = \min (S_i)$.

Assume $A \subseteq M \models T_s$ is finite and $a \in M$. Let $B = \mathrm{cl}^{(0)}(A)$, $A_0 = B \cap P^M_{\eta_0}$ and for every $1 \le i < m$ let $A_i = \mathrm{cl}^{(k)}(A) \cap P^M_{S_i}$.

As in the proof of Claim 3.21, the data below suffice to determine the k-isomorphism type of a over A. In every stage there is a polynomial bound (in terms of $|A|$) on the number of possible answers.

- Is it true that $a \in B$? If yes, then determine which element (after choosing an enumeration of B as in Claim 3.21).
- Is there some $\eta \in S$ such that $a \in P^M_\eta$? If not, we are done. If yes which is it?
- If $\eta \in S_i$ for some $1 \le i \le m$, then what is $\mathrm{tp}^{S_i}_k (a/A_i)$? Since $|A_i|$ is polynomially bounded in terms of $|A|$, the number of types is polynomially bounded.

  This information is enough in this case, since if $\mathrm{tp}^{S_i}_k (a/A_i) = \mathrm{tp}^{S_i}_k (b/A_i)$, then $a \equiv_{A_i,k} b$ (by Claim 3.20), so there is an isomorphism $f' : \mathrm{cl}^{(k)}(A_i a) \to \mathrm{cl}^{(k)}(A_i b)$ taking a to b and fixing A_i. Define $f : \mathrm{cl}^{(k)}(Aa) \to \mathrm{cl}^{(k)}(Ab)$ by

  $$\left(f' \restriction \mathrm{cl}^{(k)}(Aa) \cap P^M_{S_i}\right) \cup \left(\mathrm{id} \restriction \mathrm{cl}^{(k)}(A)\right).$$

  This is an isomorphism.
- If $\eta = \eta_0$, then the following data will suffice:
  - $\mathrm{tp}^{S_0}_k (a/A_0)$.
  - An enumeration $\langle t_l(x) \mid l < K \rangle$ (perhaps with repetitions) of terms of successor rank ≤ 1 over A_0 such that $\{t_l(a) \mid l < K\}$ lists all the new elements in $\mathrm{cl}_{\mathrm{suc}}(F)$ where $F = \mathrm{cl}^{(0)}(A_0 a)$ (exactly as in the proof of Lemma 3.15, and see below).
  - $\mathrm{tp}^{S_i}_k \left(\left\langle a^i_l \mid l < K \right\rangle / A_i\right)$ where $a^i_l = G_{\eta_0,\eta_i}(t_l(a))$ for $l < K$. By induction and by Claim 3.22, there is a polynomially bound in $|A_i|$ of the number of possible types (and $|A_i|$ itself is polynomially bounded in $|A|$).

  Explanation:

  As in the proof of Lemma 3.15, we say that an element of $\mathrm{cl}^{(k)}(A_0 a) \cap P^M_{\eta_0} = \mathrm{cl}^{(k)}_{\mathrm{suc}}(F)$ is *new* if it is a successor and not in $\mathrm{cl}^{(k)}(A_0) \cap P^M_{\eta_0}$. If e is a new element in $\mathrm{cl}^{(i)}_{\mathrm{suc}}(F)$ for $1 < i$, then, by the fact that G_{η_0,η_i} is regressive, there is $e' \in \mathrm{cl}_{\mathrm{suc}}(F)$ such that $G_{\eta_0,\eta_i}(e) = G_{\eta_0,\eta_i}(e')$. By that proof, there are at most K new elements in $\mathrm{cl}_{\mathrm{suc}}(F)$ for some constant K.

Assume $b \in M$ gives the same information listed above. Then $b \equiv^{S_0}_{A_0,k} a$ and let $f_0 : \mathrm{cl}^{(k)}_{S_0}(aA_0) \to \mathrm{cl}^{(k)}_{S_0}(bA_0)$ be a witness of that. We also know that $f_0(t_l(a)) = t_l(b)$ and that there are isomorphisms $f_i : \mathrm{cl}^{(k)}\left(\left\langle a^i_l \,|\, l < K\right\rangle A_i\right) \to \mathrm{cl}^{(k)}\left(\left\langle b^i_l \,|\, l < K\right\rangle A_i\right)$ fixing A_i (where $b^i_l = G_{\eta_0,\eta_i}(t_l(b))$). Then

$$\mathrm{id} \restriction \mathrm{cl}^{(k)}(A) \cup f_0 \cup \bigcup_{1 \le i < m} \left(f_i \restriction \mathrm{cl}^{(k)}(Aa) \cap P^M_{S_i}\right)$$

is the required isomorphism.

$\square$

The following remark implies that the theory T_S is not just dependent but strongly dependent and more. We shall come back to it in Section 6.

Remark 3.24. By closely inspecting the proof, we see that the polynomial we get as a bound for the number of Δ-types over a finite set A in terms of $|A|$ has the property that its degree is independent of Δ (but depends on S and the number of variables). More formally, the degree of the polynomial $f_{n,k}$ that bounds the number of $\equiv_{A,k}$ classes in M^n in terms of $|A|$ does not depend on k. To see that, note that this is true for $f_{1,k}$ and $|S| \le 1$ (by the proof of Claim 3.21), and continue by induction on $|S|$ (by the proof of Corollary 3.23) and n. Note that by the quantifier elimination lemma (Lemma 3.15), one can choose $f_{n+1,k}(x) = f_{1,k'}(x) \cdot f_{n,k}(x+1)$ for some $k' \ge k$.

Now we stop assuming that S is finite.

Corollary 3.25. *If $M \models T^\forall_S$ then since*

$$Th(M) = \bigcup \left\{Th\left(M \restriction L_{S_0}\right) \,|\, S_0 \subseteq S \,\&\, |S_0| < \aleph_0\right\},$$

by Remark 3.4, Corollary 3.17 is true in the case where S is infinite. So T_S is well defined in this case as well and it is in fact $\bigcup \left\{T_{S_0} \,|\, S_0 \subseteq S \,\&\, |S_0| < \aleph_0\right\}$. It eliminates quantifiers and is dependent.

Adding Constants.

We want to find an example from every cardinality, and so we add constants to the language. For a cardinal θ, the theory T^θ_S will be T_S augmented with the quantifier free diagram of a model of $T^\forall_S$ of cardinality θ. Formally:

Definition 3.26. For a cardinal θ, let $L^\theta_S = L_S \cup \{c_{\eta,i} \,|\, i < \theta, \eta \in S\}$ where $\{c_{\eta,i}\}$ are new constants. Let $T^{\forall,\theta}_S$ be the theory $T^\forall_S$ with the axioms stating that for all $\eta, \eta_1, \eta_2 \in S$ and $i, j, i'j' < \theta$ such that $\eta_1 <_{\mathrm{suc}} \eta_2$,

- $c_{\eta,i} \in P_\eta$,

- $i \neq j \Rightarrow c_{\eta,i} \neq c_{\eta,j}$,
- $i \neq j, i' \neq j' \Rightarrow c_{\eta,i} \wedge_\eta c_{\eta,j} = c_{\eta,i'} \wedge_\eta c_{\eta,j'}$,
- $\eta_1 <_{\text{suc}} \eta_2 \Rightarrow G_{\eta_1,\eta_2}(c_{\eta_1,i}) = c_{\eta_2,i}$,
- $\lim_\eta (c_{\eta,i} \wedge c_{\eta,j}) = c_{\eta,i} \wedge c_{\eta,j}$ and
- $\text{suc}_\eta (c_{\eta,i} \wedge c_{\eta,j}, c_{\eta,i}) = c_{\eta,i}$.

Corollary 3.27. *$T_S^{\forall,\theta}$ has a model completion — T_S^θ — that is complete, dependent and has quantifier elimination. This means that $T_S^{\forall,\theta}$ has JEP and the amalgamation property (AP).*

Moreover, given any model $M \models T_S^\forall$, there is a model $M' \models T_S^{\forall,\theta}$ satisfying $M' \restriction L_S \supseteq M$.

Proof. Let $T_S^\theta = T_S \cup T_S^{\forall,\theta}$. Since we only added constants, T_S^θ is dependent and has quantifier elimination. It is complete since the axioms specify exactly what is the quantifier free diagram of the structure generated by the constants.

The fact that $T_S^{\forall,\theta}$ has JEP and AP now follows from the fact that any model of it can be extended to an existentially closed one.

The "moreover" follows from JEP for $T_S^\forall$. □

4. The main theorem

From now on set $S = {}^{\omega>}2$. Here is the main theorem:

Theorem 4.1. *For every cardinal θ, T_S^θ satisfies: for every cardinal κ and limit ordinal δ, $\kappa \to (\delta)_{T,1}$ iff $\kappa \to (\delta)_\theta^{<\omega}$.*

Let us get rid of the easy direction first:

Proposition 4.2. *If $\kappa \to (\delta)_\theta^{<\omega}$ then for every theory T of cardinality $|T| \leq \theta$, $\kappa \to (\delta)_{T,1}$.*

Proof. Let $\langle a_i \,|\, i < \kappa \rangle$ be a sequence of elements in a model $M \models T$. Define $c : [\kappa]^{<\omega} \to L(T) \cup \{0\}$ as follows:

Given an increasing sequence $\eta \in \kappa^{<\omega}$, if $\lg(\eta)$ is odd, then $c(\eta) = 0$. If not, assume it is $2k$ and that $\eta = \langle \alpha_i \,|\, i < 2k \rangle$. If $a_{\alpha_0} \dots a_{\alpha_{k-1}} \equiv a_{\alpha_k} \dots a_{\alpha_{2k-1}}$ then $c(\eta) = 0$. If not there is a formula $\varphi(x_0, \dots, x_{k-1})$ such that $M \models \varphi(a_{\alpha_0}, \dots, a_{\alpha_{k-1}}) \wedge \neg\varphi(a_{\alpha_k}, \dots, a_{\alpha_{2k-1}})$, so choose such a φ and define $c(\eta) = \varphi$. By assumption there is a sub-sequence $\langle a_{\alpha_i} \,|\, i < \delta \rangle$ on which c is homogeneous. Without loss, assume that $\alpha_i = i$ for $i < \delta$.

It follows that $\langle a_i \,|\, i < \delta \rangle$ is an indiscernible sequence:

Assume not, and that $i_0 < i_1 < \dots < i_{2k-1} < \delta$ and $a_{i_0} \dots a_{i_{k-1}} \not\equiv a_{i_k} \dots a_{i_{2k-1}}$.

Then there is a formula φ such that $c(\langle i_k, \dots, i_{3k-1} \rangle) = c(\langle i_0, \dots, i_{2k-1} \rangle) = \varphi$, meaning that

$$M \models \varphi(a_{i_0}, \dots, a_{i_{k-1}}) \wedge \neg\varphi(a_{i_k}, \dots, a_{i_{2k-1}})$$

and

$$M \models \varphi\left(a_{i_k}, \ldots, a_{i_{2k-1}}\right) \wedge \neg\varphi\left(a_{i_{2k}}, \ldots, a_{i_{3k-1}}\right)$$

— a contradiction.

Let $j < \delta$ be greater than k and i_k. Then $a_{i_0} \ldots a_{i_{k-1}} \equiv a_j \ldots a_{j+k-1} \equiv a_0 \ldots a_{k-1}$ and we are done. $\square$

Before starting with the difficult direction let us introduce some notation:

Notation 4.3. Assume $M \models T_S^{\forall}$ and $x, y \in P_\eta^M$.

(1) When we say indiscernible, we shall always mean indiscernible for quantifier free formulas.
(2) We say that $x \equiv 0 \pmod{\omega}$ when $x = \lim(x)$. For $n < \omega$, we say that $x \equiv n+1 \pmod{\omega}$ where $x \neq \lim_\eta(x)$ and $\mathrm{pre}_\eta(x) \equiv n \pmod{\omega}$. Note that for a fixed n, the set $\{x \mid x \equiv n \pmod{\omega}\}$ is quantifier free definable. In addition, if the trees are well ordered as in Example 3.3, then for every x there is some $n < \omega$ such that $x \equiv n \pmod{\omega}$ (where n is the unique number satisfying $\mathrm{lev}(x) = \alpha + n$ for a limit ordinal α).
(3) Say that $x \equiv y \pmod{\omega}$ if there is $n < \omega$ such that $x \equiv n \pmod{\omega}$ and $y \equiv n \pmod{\omega}$.

The proof uses the following construction:

($\star$) Assume $S' \subseteq S$ is such that $\nu \in S' \Rightarrow \nu \restriction k \in S'$ for every $k \leq \lg(\nu)$. Assume $N \models T_{S'}^{\forall,\theta}$ and that for every $\nu \in S'$, if $\nu\hat{\ }\langle\varepsilon\rangle \notin S'$ for $\varepsilon \in \{0,1\}$, we have a model $M_\nu^\varepsilon \models T_S^{\forall,\theta}$. We may assume all models are disjoint. We build a model $M \models T_S^{\forall,\theta}$ such that $M \restriction L_{S'}^\theta \supseteq N$ and: for every $\nu \in S'$ and $\varepsilon \in \{0,1\}$ such that $\nu\hat{\ }\langle\varepsilon\rangle \notin S'$ and for every $\eta \in S$, $P^M_{\nu\hat{\ }\langle\varepsilon\rangle\hat{\ }\eta} = P_\eta^{M_\nu^\varepsilon}$. In general, for every symbol R_η from L_S^θ, let $R^M_{\nu\hat{\ }\langle\varepsilon\rangle\hat{\ }\eta} = R_\eta^{M_\nu^\varepsilon}$. For instance, $c^M_{\nu\hat{\ }\langle\varepsilon\rangle\hat{\ }\eta,i} = c_\eta^{M_\nu^\varepsilon}$ and $G^M_{\nu\hat{\ }\langle\varepsilon\rangle\hat{\ }\eta_1,\nu\hat{\ }\langle\varepsilon\rangle\hat{\ }\eta_2} = G^{M_\nu^\varepsilon}_{\eta_1,\eta_2}$ for $\eta_1 <_{\mathrm{suc}} \eta_2$.
The last thing that remains to be defined is $G^M_{\nu,\nu\hat{\ }\langle\varepsilon\rangle}$. After we have defined it, M is a model. Moreover, for every tuple $\bar{a} \in M_\nu^\varepsilon$ and for every quantifier free formula φ over L_S^θ, there is a formula φ' generated by concatenating $\nu\hat{\ }\langle\varepsilon\rangle$ to every symbol appearing in φ such that $M_\nu^\varepsilon \models \varphi(\bar{a})$ iff $M \models \varphi'(\bar{a})$. In particular, if $I \subseteq M_\nu^\varepsilon$ is an indiscernible sequence in M, it is also such in M_ν^ε.

Proof. (of the main theorem) We shall prove the following: for every cardinal κ and limit ordinal δ such that $\kappa \nrightarrow (\delta)_\theta^{<\omega}$, there is a model $M \models T_S^{\forall,\theta}$ and a set $A \subseteq P_{\langle\rangle}^M$ of size $|A| \geq \kappa$ with no non-constant indiscernible sequence in ${}^\delta(A)$. That will suffice.

The proof is by induction on κ. Note that if $\kappa \nrightarrow (\delta)_\theta^{<\omega}$ then also $\lambda \nrightarrow (\delta)_\theta^{<\omega}$ for $\lambda < \kappa$. The case analysis for some of the cases is very similar to the one done in [KS12], but we repeat it for completeness.

Divide into cases:

Case 1. $\kappa \leq \theta$. Let $M \models T_S^{\forall,\theta}$ be any model and $A = \left\{ c^M_{\langle\rangle,i} \,|\, i < \theta \right\}$.

Case 2. κ is singular. Assume that $\kappa = \bigcup \{ \lambda_i \,|\, i < \sigma \}$ where $\sigma < \kappa$ and $\lambda_i < \kappa$ for all $i < \sigma$.

Assume that N_0, A_0 are the model and set given by the induction hypothesis for σ. For all $i < \sigma$, let M_i, B_i the models and sets guaranteed by the induction hypothesis for λ_i. Let N_1 be a model of $T_S^{\forall,\theta}$ containing M_i as substructures for all $i < \sigma$ (it exists by JEP) and $A_1 = \bigcup \{ B_i \,|\, i < \sigma \}$.

Assume that $\{ a_i \,|\, i < \sigma \} \subseteq A_0$ and that $\{ b_j \,|\, \bigcup \{ \lambda_l \,|\, l < i \} \leq j < \lambda_i \} \subseteq B_i$ are enumerations witnessing that $|A_0| \geq \sigma$, $|B_i| \geq \lambda_i \backslash \bigcup \{ \lambda_l \,|\, l < i \}$.

Define $M' \models T^{\forall}_{\{\langle\rangle\}}$ as follows: $P^{M'}_{\langle\rangle} = \kappa$, $<^{M'}_{\langle\rangle} = \in$, and this is a model as in Example 3.3.

Let $N \models T^{\forall,\theta}_{\{\langle\rangle\}}$ be a model such that $N \restriction L_{\{\langle\rangle\}} \supseteq M'$. Use $(\star)$ with $M^0_{\langle\rangle} = N_0$ and $M^1_{\langle\rangle} = N_1$, and define the functions $G_{\langle\rangle,\langle 0\rangle}$ and $G_{\langle\rangle,\langle 1\rangle}$ as follows: For a limit $\alpha < \kappa$ and $0 < n < \omega$, define $G^M_{\langle\rangle,\langle 0\rangle}(\alpha + n) = a_{\min\{ j < \sigma |\, \alpha < \lambda_j \}}$ and $G^M_{\langle\rangle,\langle 1\rangle}(\alpha + n) = b_\alpha$.

Let $A = \kappa = P^{M'}_{\langle 0\rangle}$. Assume that $\langle \beta_i \,|\, i < \delta \rangle$ is an indiscernible sequence contained in A.

Obviously it cannot be that $\beta_1 < \beta_0$. Assume that $\beta_0 < \beta_1$. There are limit ordinals α_i and natural number n_i such that $\beta_i = \alpha_i + n_i$, i.e. $\beta_i \equiv n_i \pmod{\omega}$. By indiscernibility, n_i is constant, and denote it by n. So $\langle \mathrm{suc}(\beta_{2i}, \beta_{2i+1}) = \beta_{2i} + 1 \,|\, i < \delta \rangle$ is an indiscernible sequence of successor ordinals.

$\left\langle G_{\langle\rangle,\langle 0\rangle}(\beta_{2i} + 1) \,|\, i < \delta \right\rangle$ must be constant by the choice of A_0, and assume it is a_{i_0} for $i_0 < \sigma$. It follows that $\alpha_{2i} \in \lambda_{i_0} \backslash \bigcup \{ \lambda_l \,|\, l < i_0 \}$. This means that $G_{\langle\rangle,\langle 1\rangle}(\beta_{2i} + 1) = b_{\alpha_{2i}} \subseteq B_{i_0}$ for all $i < \sigma$, and so α_{2i} must be constant. This means that $\langle \beta_{2i} \,|\, i < \delta \rangle$ is constant — a contradiction.

Case 3. κ is regular but not strongly inaccessible. Then there is some $\lambda < \kappa$ such that $2^\lambda \geq \kappa$.

Let $M_0 \models T_S^{\forall,\theta}$ and $A_0 \subseteq P^{M_0}_{\langle\rangle}$ be from the induction hypothesis for λ. Assume that $A_0 \supseteq \{ a_i \,|\, i \leq \lambda \}$ where $a_i \neq a_j$ for $i \neq j$.

Let $M' \models T^{\forall}_{\{\langle\rangle\}}$ be such that $P^{M'}_{\langle\rangle} = {}^{\lambda \geq} 2$ ordered by first segment. We give it a structure of $T^{\forall}_{\{\langle\rangle\}}$ as in Example 3.3.

Let $N \models T^{\forall,\theta}_{\{\langle\rangle\}}$ be any model such that $N \restriction L_{\{\langle\rangle\}} \supseteq M'$. We use the construction $(\star)$ to build a model $M \models T^{\forall}_S$ using N and $M^0_{\langle\rangle} = M^1_{\langle\rangle} = M_0$: we need to define the functions $G_{\langle\rangle,\langle 0\rangle}$ and $G_{\langle\rangle,\langle 1\rangle}$:

For $f \in P^{M'}_{\langle\rangle}$ such that $\lg(f) = \alpha + n$ for some limit α and $n < \omega$, define $G^{M'}_{\langle\rangle,\langle 0\rangle}(f) = a_\alpha$. There are no further limitations on the functions $G^{M'}_{\langle\rangle,\langle 0\rangle}$ and $G^{M'}_{\langle\rangle,\langle 1\rangle}$ as long as they are regressive.

Let $A = {}^{\lambda \geq} 2 = P^{M'}_{\langle\rangle}$. Assume for contradiction that $\langle f_i \,|\, i < \delta \rangle$ is a non-constant indiscernible sequence contained in A.

It cannot be that $f_1 < f_0$, because by indiscernibility, we would have an infinite decreasing sequence.

It cannot be that $f_0 < f_1$: In that case, $\langle f_i \mid i < \delta \rangle$ is increasing. For all $i < \delta$, let $g_i = \mathrm{suc}(f_{2i}, f_{2i+1})$. The sequence $\langle g_i \mid i < \delta \rangle$ is an indiscernible sequence contained in $\mathrm{Suc}\left(P_{\langle\rangle}\right)$ and so $g_i \equiv n \pmod{\omega}$ for some constant $n < \omega$. Hence $\langle \lg(g_i) - n \mid i < \delta \rangle$ is increasing and $\left\langle G_{\langle\rangle,\langle 0\rangle}(g_i) = a_{\lg(g_i)-n} \mid i < \delta \right\rangle$ is a non-constant indiscernible sequence contained in A_0 — a contradiction.

Denote $h_i = f_0 \wedge f_{i+1}$ for $i < \delta$. This is an indiscernible sequence, and by the same arguments, it cannot decrease or increase. But since $h_i < f_0$, it follows that h_i is constant.

Assume that $f_0 \wedge f_1 < f_1 \wedge f_2$, then $f_1 \wedge f_2 < f_2 \wedge f_3$ and so $f_{2i} \wedge f_{2i+1} < f_{2(i+1)} \wedge f_{2(i+1)+1}$ for all $i < \delta$, and again — $\langle f_{2i} \wedge f_{2i+1} \rangle$ is an increasing indiscernible sequence — we reach a contradiction.

Similarly, it cannot be that $f_0 \wedge f_1 > f_1 \wedge f_2$. As both sides are smaller or equal to f_1, it must be that

$$f_0 \wedge f_2 = f_0 \wedge f_1 = f_1 \wedge f_2.$$

But this is a contradiction (because if $\alpha = \lg(f_0 \wedge f_1)$ then $|\{f_0(\alpha), f_1(\alpha), f_2(\alpha)\}| = 3$, but the range of the functions is $\{0, 1\}$).

Case 4. κ is strongly inaccessible. With this case we shall deal in the next section.

□

5. The inaccessible case

Assumption 5.1. *Assume from now on that $\theta < \kappa$ is strongly inaccessible and that $\kappa \nrightarrow (\delta)^{<\omega}_{\theta}$.*

This means that for all $\lambda < \kappa$, $\lambda \nrightarrow (\delta)^{<\omega}_{\theta}$, so we may use the induction hypothesis in the proof.

First let us define an abstract class of structures and analyze indiscernible sequences in it. Then we shall apply the analysis to a specific structure. This structure will be used when we return to the proof.

Again, throughout this section, indiscernible means "indiscernible for quantifier free formulas".

Analysis of indiscernibles in $\mathcal{T}$.

We introduce a class of models of $T^{\forall}_{\omega}$, and then we shall analyze indiscernible sequences in them.

Definition 5.2. Let $\mathcal{T}$ be the class of models $M \models T^{\forall}_{\omega}$ that satisfy

(1) For all $n < \omega$, $(P_n, <)$ is a standard tree, and we give it structure as in Example 3.3.
(2) For $t \in P_n$, $\mathrm{lev}(G_{n,n+1}(t)) \leq \mathrm{lev}(t)$. As here $S = \omega$, we denote $G_{n,n+1}$ by G_n.
(3) $G_n : \mathrm{Suc}(P_n) \to \mathrm{Suc}(P_n)$ (i.e. we demand that the image is also a successor).

(4) If $\langle s_i \,|\, i < \delta \rangle$ is an increasing sequence in $\mathrm{Suc}\,(P_n)$ such that $s_i \equiv s_j \pmod{\omega}$ for all $i < j < \delta$ then $i < j \Rightarrow G_n\,(s_i) \neq G_n\,(s_j)$.

Notation 5.3. For $M \in \mathcal{T}$,

(1) We say that $s, t \in P_n$ are neighbors, denoted by $tE^{\mathrm{nb}}s$ when $\{x \,|\, x < t\} = \{x \,|\, x < s\}$. This is an equivalence relation. As P_n is a normal tree, for t of a limit level its E^{nb}-class is $\{t\}$.
(2) For all $M \in \mathcal{T}$, let $\mathrm{Suc}\,(M) = \bigcup \left\{\mathrm{Suc}\left(P_n^M\right) \,|\, n < \omega\right\}$.

We need the following generalization of indiscernible sequences for $\mathcal{T}$:

Definition 5.4. A sequence $\bar{s} = \langle s_i \,|\, i < \delta \rangle$ is called *Nearly Indiscernible* (in short *NI*) if

(1) There is $n < \omega$ such that for all $\delta^* \leq \delta$ every sub-sequence $\left\langle s_{i_j} \,|\, j < \delta^* \right\rangle$ such that $i_j + n \leq i_{j+1}$ is indiscernible with the same quantifier free type. We call this property "the sub-sequence property".
(2) For $i, j < \delta$ and $k < \omega$, $\mathrm{tp}_{\mathrm{qf}}\,(s_i, \ldots, s_{i+k}) = \mathrm{tp}_{\mathrm{qf}}\,(s_j, \ldots, s_{j+k})$. We call this property "sequential homogeneity".

Definition 5.5. A sequence $\bar{s} = \langle s_i \,|\, i < \delta \rangle$ is called *Hereditarily Nearly Indiscernible* (in short *HNI*) if:

For every term $\sigma\,(x_0, \ldots, x_{n-1})$, the sequence $\bar{t} = \langle t_i \,|\, i < \delta \rangle$ defined by $t_i = \sigma\,(s_i, \ldots, s_{i+n-1})$ is NI.

Example 5.6. If $\bar{s} = \langle s_i \,|\, i < \delta \rangle$ is indiscernible, then it is HNI.

Proof. If $t_i = \sigma\,(s_i, \ldots, s_{i+n-1})$, then any sub-sequence of $\bar{t}$ where the distance between two consecutive elements is at least n is an indiscernible sequence (with a constant quantifier free type).

Note that for a quantifier free formula φ,

$$\varphi\,(t_i, \ldots, t_{i+k}) = \varphi\,(\sigma\,(s_i, \ldots, s_{i+n-1})\,, \ldots, \sigma\,(s_{i+k}, \ldots, s_{i+n+k-1}))\,.$$

Let $i, j < \delta$. As $\mathrm{tp}_{\mathrm{qf}}\,(s_i, \ldots, s_{i+n+k-1}) = \mathrm{tp}_{\mathrm{qf}}\,(s_j, \ldots, s_{j+n+k-1})$, it follows that

$$\mathrm{tp}_{\mathrm{qf}}\,(t_i, \ldots, t_{i+k}) = \mathrm{tp}_{\mathrm{qf}}\,(t_j, \ldots, t_{j+k})\,.$$

□

Notation 5.7.

(1) $\bar{s}$ and $\bar{t}$ will denote δ-sequences, e.g. $\bar{s} = \langle s_i \,|\, i < \delta \rangle$.
(2) If there is some $n < \omega$ such that $\bar{s}$ is contained in P_n^M, then we write $<$ instead of $<_n$ etc.

Definition 5.8. Assume $M \in \mathcal{T}$.

(1) $\mathrm{ind}(M)$ is the set of all non-constant indiscernible sequences $\bar{s} \in {}^{\delta}\mathrm{Suc}(M)$.
(2) $\mathrm{HNind}(M)$ is the set of all non-constant HNI sequences $\bar{s} \in {}^{\delta}\mathrm{Suc}(M)$.
(3) $\mathrm{ai}(M)$ is the set of sequences such that there exists some $n < \omega$ with $\bar{s} \in {}^{\delta}\left(P_n^M\right)$ and $s_i \wedge s_{i+1} < s_{i+1} \wedge s_{i+2}$ (ai means almost increasing, note that if $\bar{s}$ is increasing then it is here).
(4) $\mathrm{ind}_f(M)$ is the set of all sequences $\bar{s} \in \mathrm{ind}(M)$ such that $s_i \wedge s_j$ is constant for all $i < j < \delta$ (f comes from fan).
(5) $\mathrm{ind}_i(M)$ is the set of all increasing sequences $\bar{s} \in \mathrm{ind}(M)$.
(6) $\mathrm{ind}_{\mathrm{ai}}(M) = \mathrm{ind}(M) \cap \mathrm{ai}(M)$.
(7) Define $\mathrm{HNind}_f(M)$, $\mathrm{HNind}_i(M)$ and $\mathrm{HNind}_{\mathrm{ai}}(M)$ in the same way, but we demand that the sequences are HNI.

From now on, assume $M \in \mathcal{T}$.

Remark 5.9. If $\bar{s} \in \mathrm{ai}(M)$, then $s_i \wedge s_{i+n} = s_i \wedge s_{i+1}$ for all $2 \leq n < \omega$ and $i < \delta$ (prove by induction on n).

Proposition 5.10. $\mathrm{HNind}(M) = \mathrm{HNind}_{\mathrm{ai}}(M) \cup \mathrm{HNind}_f(M)$.

Proof. Assume that $\bar{s} \in \mathrm{HNind}(M)$. Since $\bar{s}$ is NI, there is some $n < \omega$ that witnesses the subsequence property. As for $i < j < k$, $s_i \wedge s_j$ is comparable with $s_j \wedge s_k$, by Ramsey, there is an infinite subset $A \subseteq \omega$ that satisfies one of the following possibilities:

(1) For all $i < j < k \in A$, $s_i \wedge s_j = s_j \wedge s_k$, or
(2) For all $i < j < k \in A$, $s_i \wedge s_j < s_j \wedge s_k$.

(note that it cannot be that $s_j \wedge s_k < s_i \wedge s_j$ because the trees are well ordered).

Assume (1) is true.

It follows that if $i < j < k < l \in A$ then $s_i \wedge s_j = s_j \wedge s_k = s_k \wedge s_l$. If $n \leq j-i, k-j, l-k$, then by the choice of n, the same is true for all $i < j < k < l < \delta$ where the distances are at least n. Moreover, given $i < j, k < l$ such that $n \leq j - i$ and $n \leq l - k$, then $s_i \wedge s_j = s_{\max\{j,l\}+n} \wedge s_{\max\{j,l\}+2n}$, and the same is true for $s_k \wedge s_l$. It follows that $s_i \wedge s_j = s_k \wedge s_l$.

Choose some $0 < i < n$.

Assume for contradiction that $s_0 \wedge s_i < s_i \wedge s_{2i}$, then by sequential homogeneity $\langle s_{i\alpha} \,|\, \alpha < \delta \rangle \in \mathrm{ai}(M)$. In this case, by Remark 5.9, $s_0 \wedge s_i < s_i \wedge s_{2i} = s_i \wedge s_{ni+i}$. But $s_0 \wedge s_{ni+i} = s_i \wedge s_{ni+i}$, and so on the one hand $s_0 \wedge s_i < s_0 \wedge s_{ni+i}$, and on the other $s_0 \wedge s_{ni+i} \leq s_i$ — together it's a contradiction.

It cannot be that $s_0 \wedge s_i > s_i \wedge s_{2i}$ since the trees are well ordered.

So (again by the sequential homogeneity) it must be that $s_0 \wedge s_i = s_i \wedge s_{2i} = \ldots = s_{ni} \wedge s_{ni+i}$. So necessarily $s_0 \wedge s_i \leq s_0 \wedge s_{ni}$, but in addition $s_0 \wedge s_{ni} = s_0 \wedge s_{ni+i}$ (since the distance is at least n) and so $s_0 \wedge s_i = s_{ni} \wedge s_{ni+i} \geq s_0 \wedge s_{ni}$, and hence $s_0 \wedge s_i = s_0 \wedge s_{ni} = s_0 \wedge s_n$.

It follows that $s_{i_0} \wedge s_{i_0+i} = s_{i_0} \wedge s_{i_0+n} = s_0 \wedge s_n$ for every $i_0 < \delta$. This is true for all i such that $i_0 + i < \delta$ and so $s_i \wedge s_j = s_0 \wedge s_n$ for all $i < j < \delta$. So in this case $\bar{s} \in \mathrm{HNind}_f(M)$.

Assume (2) is true. Assume that $i < j < k \in A$ and the distances are at least n. Then, as $s_i \wedge s_j < s_j \wedge s_k$, it follows from the sub-sequence property that $\langle s_{n\alpha} \mid \alpha < \delta \rangle \in \mathrm{ai}(M)$ and that $\langle s_0, s_{n+1}, s_{3n}, s_{4n}, \ldots \rangle \in \mathrm{ai}(M)$. In particular, by Remark 5.9, $s_0 \wedge s_n = s_0 \wedge s_{3n} = s_0 \wedge s_{n+1}$.

If $s_0 \wedge s_1 < s_1 \wedge s_2$, then $\bar{s} \in \mathrm{HNind}_{\mathrm{ai}}(M)$ by sequential homogeneity and we are done, so assume this is not the case.

It cannot be that $s_0 \wedge s_1 > s_1 \wedge s_2$ (because the trees are well ordered).

Assume for contradiction that $s_0 \wedge s_1 = s_1 \wedge s_2$. By sequential homogeneity it follows that $s_0 \wedge s_1 = s_n \wedge s_{n+1}$. We also know that $s_0 \wedge s_n = s_0 \wedge s_{n+1}$, and together we have $s_0 \wedge s_1 = s_0 \wedge s_{n+1}$, and again by sequential homogeneity, $s_n \wedge s_{2n+1} = s_n \wedge s_{n+1}$, and so $s_n \wedge s_{2n+1} = s_0 \wedge s_n$ — a contradiction (because the distances are at least n). □

Definition 5.11. Define the function $H : \mathrm{HNind}_{\mathrm{ai}}(M) \to \mathrm{HNind}(M)$ as follows: given $\bar{s} \in \mathrm{HNind}_{\mathrm{ai}}(M)$, let $H(\bar{s}) = \bar{t}$ where $t_i = G(\mathrm{suc}(\lim(s_i \wedge s_{i+1}), s_{i+1}))$.

Remark 5.12. H is well defined: if $\bar{s} \in \mathrm{HNind}_{\mathrm{ai}}(M)$ then $H(\bar{s})$ is in $\mathrm{HNind}(M)$. This is because $\bar{t} = H(\bar{s})$ is not constant — by clause (4) of Definition 5.2 (it is applicable: the sequence $\langle s_i \wedge s_{i+1} \mid i < \delta \rangle$ is NI and increasing, so there is some $n < \omega$ such that $s_i \wedge s_{i+1} \equiv n \pmod{\omega}$ for all $i < \delta$, and hence $\langle \lim(s_i \wedge s_{i+1}) \mid i < \delta \rangle$ is increasing).

Corollary 5.13. *Let $\bar{s} \in \mathrm{HNind}_{\mathrm{ai}}(M)$. If for no $n < \omega$, $H^{(n)}(\bar{s}) \in \mathrm{HNind}_f(M)$, then for all $n < \omega$, $H^{(n)}(\bar{s}) \in \mathrm{HNind}_{\mathrm{ai}}(M)$. Moreover, in this case there exists some $K < \omega$ such that for all $n \geq K$, if $\bar{t} = H^{(n)}(\bar{s})$ then* $\mathrm{suc}(\lim(t_i \wedge t_{i+1}), t_i) = t_i$.

Proof. By Proposition 5.10, it follows by induction on $n < \omega$ that $H^{(n)}(\bar{s}) \in \mathrm{HNind}_{\mathrm{ai}}(M)$ and so $H^{(n+1)}(\bar{s})$ is well defined.

For $n < \omega$, let $\bar{s}_n = H^{(n)}(\bar{s})$, and let us enumerate this sequence by $\bar{s}_n = \langle s_{n,i} \mid i < \delta \rangle$.

$\mathrm{lev}(\lim(s_{n,0} \wedge s_{n,1})) < \mathrm{lev}(s_{n,0})$ because $\mathrm{lev}(s_{n,0})$ is a successor ordinal (by clause (3) of Definition 5.2) while $\mathrm{lev}(\lim(x))$ is a limit for all $x \in M$.

So $\mathrm{lev}(\mathrm{suc}(\lim(s_{n,0} \wedge s_{n,1})), s_{n,1}) \leq \mathrm{lev}(s_{n,0})$, and so by clause (2) of Definition 5.2,

$$\langle \mathrm{lev}(s_{n,0}) \mid n < \omega \rangle$$

is a $\leq$-decreasing sequence.

Hence there is some $K < \omega$ and some α such that $\mathrm{lev}(s_{n,0}) = \alpha$ for all $K \leq n$. Assume without loss of generality that $K = 0$.

Let $n < \omega$. We know that

$$\begin{aligned} \operatorname{lev}(s_{n+1,0}) \leq \operatorname{lev}(\operatorname{suc}(\lim(s_{n,0} \wedge s_{n,1}), s_{n,1})) &= \\ &= \operatorname{lev}(\operatorname{suc}(\lim(s_{n,0} \wedge s_{n,1}), s_{n,0})) \\ &\leq \operatorname{lev}(s_{n,0}) \end{aligned}$$

But the left side and the right side are equal and $\operatorname{suc}(\lim(s_{n,0} \wedge s_{n,1}), s_{n,0}) \leq s_{n,0}$, so

$$\operatorname{suc}(\lim(s_{n,0} \wedge s_{n,1}), s_{n,0}) = s_{n,0}.$$

By sequential homogeneity, $\operatorname{suc}(\lim(s_{n,i} \wedge s_{n,i+1}), s_{n,i}) = s_{n,i}$ for all $i < \delta$ as desired. □

Constructing a model in $\mathcal{T}$.

By Assumption 5.1, we have a coloring $c : [\kappa]^{<\omega} \to \theta$ that witnesses the fact that $\kappa \nrightarrow (\delta)^{<\omega}_{\theta}$. Fix c, and also a pairing function (a bijection) $\operatorname{pr} : \theta \times \theta \to \theta$ and projections $\pi_1, \pi_2 : \theta \to \theta$ (defined by the equations $\pi_1(\operatorname{pr}(\alpha, \beta)) = \alpha$ and $\pi_2(\operatorname{pr}(\alpha, \beta)) = \beta$). For us, 0 is considered to be a limit ordinal, and for an ordinal α, let $\operatorname{Lim}(\alpha) = \{\beta < \alpha \mid \beta \text{ is a limit}\}$.

Definition 5.14. $\mathbf{P} = \mathbf{P}_{\theta,\kappa}$ is the set of triples $\mathbf{p} = (d, M, E) = (d_{\mathbf{p}}, M_{\mathbf{p}}, E_{\mathbf{p}})$ where:

(1) M is a model of $T^{\forall}_{\{\emptyset\}}$ as in Example 3.3 and $M = P^M_{\emptyset}$ (i.e. M is a standard tree). Some notation:
   (a) We write $<_{\mathbf{p}}$ instead of $<^M_{\emptyset}$ etc., or omit $\mathbf{p}$ when it is clear.
   (b) Let $\operatorname{Suc}_{\lim}(M)$ be the set of all $t \in \operatorname{Suc}(M)$ such that $\operatorname{lev}(t) - 1$ is a limit.

(2) E is an equivalence relation refining E^{nb}. Moreover, for levels that are not $\alpha + 1$ for limit α it equals E^{nb}. By normality E is equality on limit elements, so it is interesting only on $\operatorname{Suc}_{\lim}(M)$.

(3) For every E^{nb} equivalence class C, $|C/E| < \kappa$.

(4) d is a function from $\left\{\eta \in {}^{\omega>}\operatorname{Suc}_{\lim}(M) \mid \eta(0) < \ldots < \eta(\lg(\eta) - 1)\right\}$ to θ.

(5) We say that $\mathbf{p}$ is *hard* if there is no increasing sequence of elements $\bar{s}$ of length δ from $\operatorname{Suc}_{\lim}(M)$ such that:

   For all $n < \omega$ there is c_n such that for every $i_0 < \ldots < i_{n-1} < \delta$, $d\left(s_{i_0}, \ldots, s_{i_{n-1}}\right) = c_n$.

Example 5.15. κ is a standard tree. Let $\mathbf{p}_c = (c \restriction \operatorname{Suc}_{\lim}(\kappa), \kappa, =) \in \mathbf{P}$. Then $\mathbf{p}_c$ is hard.

Definition 5.16. Let $\mathbf{p} = (d, M, E) \in \mathbf{P}$, x a variable and $A \subseteq \operatorname{Suc}_{\lim}(M)$ a linearly ordered set.

(1) Say that p is a *d-type* over A if p is a consistent set of equations of the form

   $d(a_0, \ldots, a_{n-1}, x) = \varepsilon$ where $n < \omega$, $\varepsilon < \theta$ and $a_0 < \ldots < a_{n-1} \in A$.

(2) Consistency here means that p does not contain a subset of the form

$$\{d(a_0, \ldots, a_{n-1}, x) = \varepsilon, d(a_0, \ldots, a_{n-1}, x) = \varepsilon'\}$$

   for $\varepsilon \neq \varepsilon'$.

(3) Say that p is complete if for every increasing sequence $\langle a_0, \dots, a_{n-1} \rangle$ from A there is such an equation in p.

(4) If $B \subseteq A$ then for a d-type p over A, let

$$p \restriction B = \{ d(a_0, \dots, a_{n-1}, x) = \varepsilon \in p \,|\, a_0, \dots, a_{n-1} \in B \}.$$

(5) For $t \in \mathrm{Suc}_{\lim}(M)$,

$$\begin{aligned} \mathrm{dtp}(t/A) &= \{ d(a_0, \dots, a_{n-1}, x) = \varepsilon | \\ &\qquad a_0 < \dots < a_{n-1} \in A, a_{n-1} < t, d_{\mathbf{p}}(a_0, \dots, a_{n-1}, t) = \varepsilon \}. \end{aligned}$$

For an element $t \in \mathrm{Suc}_{\lim}(M)$, $t \models p$ means that t satisfies all the equations in p when we replace d by d_p.

(6) Let $R(A)$ be the set of all complete d-types over A.

Now we define the function $\mathbf{q}$ from $\mathbf{P}$ to $\mathbf{P}$.

Definition 5.17. For $\mathbf{p} \in \mathbf{P}$, define $\mathbf{q} = \mathbf{q}(\mathbf{p}) = (M_{\mathbf{q}}, d_{\mathbf{q}}, E_{\mathbf{q}}) = (M, d, E) \in \mathbf{P}$ by:

- M is the set of pairs $a = (\Gamma, \eta) = (\Gamma_a, \eta_a)$ such that:
  (1) There is $\alpha < \kappa$ such that $\eta : \alpha \to \mathrm{Suc}_{\lim}(M_{\mathbf{p}})$ and $\Gamma : \mathrm{Lim}(\alpha) \to R(M_{\mathbf{p}})$. Denote $\lg(\Gamma, \eta) = \lg(\eta) = \alpha$. If α is a successor ordinal, let $l_{(\Gamma,\eta)} = \eta(\alpha - 1) \in M_{\mathbf{p}}$.
  (2) For $\beta < \alpha$ limit, $\Gamma(\beta) \in R(\{\eta(\beta') \,|\, \beta' \leq \beta\})$.
  (3) If $0 < \alpha$ then $\eta(0) \models \Gamma(0) \restriction \emptyset$.
  (4) For $\beta' < \beta < \alpha$, $\eta(\beta') < \eta(\beta)$ (η is increasing in $M_{\mathbf{p}}$).
  (5) If $\beta' < \beta < \alpha$ are limit ordinals then $\Gamma(\beta') \subseteq \Gamma(\beta)$.
  (6) If $\beta' < \beta < \alpha$ and β' is a limit ordinal then $\eta(\beta) \models \Gamma(\beta')$.
  (7) For $\beta < \alpha$, there is no $t < \eta(\beta)$ that satisfies
    (a) $t \in \mathrm{Suc}_{\lim}(M_{\mathbf{p}})$,
    (b) $\eta(\beta') < t$ for all $\beta' < \beta$,
    (c) $t \models \Gamma(0) \restriction \emptyset$, and
    (d) $t \models \Gamma(\beta')$ for all limit $\beta' < \beta$.
  (8) The order on T is $(\Gamma, \eta) < (\Gamma', \eta')$ iff $\Gamma \triangleleft \Gamma'$ and $\eta \triangleleft \eta'$ (where $\triangleleft$ means first segment). It follows that for $a = (\Gamma, \eta)$, $\mathrm{lev}(a) = \lg(a)$.
- d is defined as follows: suppose $a_0 < \dots < a_{n-1} \in \mathrm{Suc}_{\lim}(M)$ and $a_i = (\Gamma_i, \eta_i)$.

  Let $t_i = l_{a_i} = \eta_i(\lg(a_i) - 1)$ and $p = \Gamma_{n-1}(\lg(a_{n-1}) - 1)$. Let $\varepsilon \in \theta$ be the unique color such that $d(t_0, \dots, t_{n-1}, x) = \varepsilon \in p$. Then

$$d(a_0, \dots, a_{n-1}) = \mathrm{pr}(\varepsilon, c(\mathrm{lev}(a_0), \dots, \mathrm{lev}(a_{n-1}))).$$

- E is defined as follows: $(\Gamma_1, \eta_1) E (\Gamma_3, \eta_2)$ iff
  - $\lg(\eta_1) = \lg(\eta_2)$, so equals to some $\alpha < \kappa$,

– $\eta_1 \restriction \beta = \eta_2 \restriction \beta, \Gamma_1 \restriction \beta = \Gamma_2 \restriction \beta$ for all $\beta < \alpha$ (so they are E^{nb}-equivalent),
– $\Gamma_1(0) \restriction \emptyset = \Gamma_2(0) \restriction \emptyset$, and
– If $\alpha = \beta + n$ for $\beta \in \operatorname{Lim}(\alpha)$ and $n < \omega$ then for all $\alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \beta$,

$$d(\eta_1(\alpha_0), \ldots, \eta_1(\alpha_{k-1}), \eta_1(\beta), x) = \varepsilon \in \Gamma_1(\beta) \quad \Leftrightarrow$$

$$d(\eta_2(\alpha_0), \ldots, \eta_2(\alpha_{k-1}), \eta_2(\beta), x) = \varepsilon \in \Gamma_2(\beta)$$

Note that it follows that if $1 < n$, and $(\Gamma_1, \eta_1) E^{\text{nb}} (\Gamma_2, \eta_2)$, then $\Gamma_1(\beta) = \Gamma_2(\beta)$ and $\eta_1(\beta) = \eta_2(\beta)$, so they are E-equivalent.

In the next claims we assume that $\mathbf{p} \in \mathbf{P}$ and $\mathbf{q} = \mathbf{q}(\mathbf{p})$.

Remark 5.18. $\operatorname{lev}(a) = \lg(a)$ for $a \in M_{\mathbf{q}}$ and $aE^{\text{nb}}b$ iff $\operatorname{lev}(a) = \operatorname{lev}(b)$ and $a \restriction \alpha = b \restriction \alpha$ for all $\alpha < \operatorname{lev}(a)$.

Claim 5.19. $\mathbf{q} \in \mathbf{P}_{\theta,\kappa}$ and moreover it is hard.

Proof. The fact that $M_{\mathbf{q}}$ is a standard tree is trivial. Also, E refines E^{nb} by definition.

We must show that the number of E-classes inside a given E^{nb}-class is bounded.

Given a (partial) d-type p over $M_{\mathbf{p}}$ and $t \in M_{\mathbf{p}}$, let p^t be the set of equations we get by replacing all appearances of t by a special letter $*$.

Assume that A is an E^{nb}-class contained in $\operatorname{Suc}_{\lim}(M_{\mathbf{q}})$, and that for every $a \in A$, $\operatorname{lev}(a) = \alpha+1$ where α is limit. Assume $a \in A$ and let $B = \{*\} \cup \operatorname{im}(\eta_a) \setminus \{l_a\}$ (since A is an E^{nb}-class, this set does not depend on the choice of a). Consider the map ε defined by $a \mapsto \Gamma_a(\alpha)^{l_a}$. Then, $a, b \in A$ are E equivalent iff $\varepsilon(a) = \varepsilon(b)$. Therefore this map induces an injective map from A/E to this set of types. The size of this set is at most $2^{|B|+\theta+\aleph_0}$. But $|B| = |\alpha| < \kappa$, and $\theta < \kappa$ by assumption, so $|A/E| < \kappa$ (as κ is a strong limit).

$\mathbf{q}$ is hard: if $\bar{s} = \langle s_i \mid i < \delta \rangle$ is a counterexample then $\langle \operatorname{lev}(s_i) \mid i < \delta \rangle$ contradicts the choice of c. □

Proposition 5.20.

(1) For all $a \in \operatorname{Suc}(M_{\mathbf{q}})$, $\operatorname{lev}_{M_{\mathbf{q}}}(a) \leq \operatorname{lev}_{M_{\mathbf{p}}}(l_a)$.
(2) Assume $t \in \operatorname{Suc}_{\lim}(M_{\mathbf{p}})$. Then there is some $a = (\Gamma, \eta) \in M_{\mathbf{q}}$ such that $l_a = t$.

Proof. (1) Let $\operatorname{lev}_{M_{\mathbf{q}}}(a) = \alpha$. Then $\langle \operatorname{pre}(\eta_a(\beta)) \mid \beta < \alpha \rangle$ is an increasing sequence below l_a, hence $\alpha \leq \operatorname{lev}_{M_{\mathbf{p}}}(l_a)$.

(2) We try to build $(\Gamma_\alpha, \eta_\alpha)$ by induction on $\alpha < \kappa$ so that $(\Gamma_\alpha, \eta_\alpha) \in M_{\mathbf{q}}$; $\lg(\eta_\alpha) = \alpha$; it is an increasing sequence in $<_{\mathbf{q}}$; $\eta_\alpha(\beta) < t$ for $\beta < \alpha$ and if β is a limit then $\Gamma_\alpha(\beta) = \operatorname{dtp}(t / \{\eta_\alpha(\beta') \mid \beta' \leq \beta\})$.

We must get stuck in a successor stage, i.e. there is some β such that we cannot continue the construction for $\alpha = \beta + 1$. In fact, we will get stuck at the latest in stage $\operatorname{lev}_{M_{\mathbf{p}}}(t)$ by (1).

Define $\eta = \eta_\beta \cup \{(\beta, t)\}$, $\Gamma = \Gamma_\beta$ unless β is a limit in which case let $\Gamma(\beta)$ be any complete type in x over $\{\eta(\beta') \mid \beta' \leq \beta\}$ containing $\bigcup\{\Gamma_\beta(\beta') \mid \beta' \in \mathrm{Lim}(\beta)\} \cup \{d(x) = d_{\mathbf{p}}(t)\}$.

By construction, $(\Gamma, \eta) \in M_{\mathbf{q}}$. □

Now we build a model in $\mathcal{T}$ using $\mathbf{P}$:

Definition 5.21.

(1) Define $\mathbf{p}_0 = \mathbf{p}_c$ (see Example 5.15), and for $n < \omega$, let $\mathbf{p}_{n+1} = \mathbf{q}(\mathbf{p}_n)$.
(2) Define $P_n = M_{\mathbf{p}_n}$, $d_n = d_{\mathbf{p}_n}$ and $E_n = E_{\mathbf{p}_n}$.
(3) Let $M_c = \bigcup_{n<\omega} P_n$ (we assume that the P_n's are mutually disjoint). So $P_n^{M_c} = P_n$.
(4) $M_c \models T_\omega^\forall$ when we interpret the relations in the language as they are induced from each P_n and in addition:
(5) Define $G_n^{M_c} : \mathrm{Suc}(P_n) \to \mathrm{Suc}(P_{n+1})$ as follows: let $a \in \mathrm{Suc}(P_n)$ and $a' = \mathrm{suc}(\lim(a), a)$. By the claim above, there is an element $(\Gamma, \eta)_a \in \mathrm{Suc}(P_{n+1})$ such that $l_{(\Gamma,\eta)_a} = a'$. Choose such an element for each a, and define $G_n^{M_c}(a) = (\Gamma, \eta)_a$.

Corollary 5.22. *$M_c \in \mathcal{T}$.*

Proof. All the demands of Definition 5.2 are easy. For instance, clause (2) follows from Proposition 5.20. Clause (4) follows from the fact that if $\langle s_i \mid i < \delta \rangle$ is an increasing sequence in P_n such that $s_i \equiv s_j \pmod{\omega}$ then $\langle \mathrm{suc}(\lim(s_i), s_i) \mid i < \delta \rangle$ is increasing, so $l_{G_n(s_i)} \neq l_{G_n(s_j)}$ for $i \neq j$. □

Notation 5.23. Again, we do not write the index $\mathbf{p}_n$ when it is clear from the context (for instance we write $d(s_0, \ldots, s_k)$ instead of $d_{\mathbf{p}_n}(s_0, \ldots, s_k)$).

Lemma 5.24. *Assume that $\bar{s} \in \mathrm{HNind}_{\mathrm{ai}}(M_c)$ and $\bar{t} = H(\bar{s}) \in \mathrm{HNind}_{\mathrm{ai}}(M_c)$ (see Definition 5.11) satisfy that for all $i < \delta$:*

- $\mathrm{suc}(\lim(s_i \wedge s_{i+1}), s_i) = s_i$, *and*
- $\mathrm{suc}(\lim(t_i \wedge t_{i+1}), t_i) = t_i$.

Claim 5.25. Then, if $u_i = \mathrm{suc}(\lim(s_i \wedge s_{i+1}), s_{i+1})$ and $v_i = \mathrm{suc}(\lim(t_i \wedge t_{i+1}), t_{i+1})$ for $i < \delta$, then:

(1) $\langle d(u_i) \mid 1 \leq i < \delta \rangle$ is constant.
(2) $d(u_{i_0}, \ldots, u_{i_n}) = \pi_1\left(d\left(v_{i_0}, \ldots, v_{i_{n-1}}\right)\right)$ for $1 \leq i_0 < \ldots < i_n < \delta$ (recall that π_1 is defined by $\pi_1(\mathrm{pr}(i,j)) = i$).

Proof. (1) By definition, $t_i = G(u_i)$. Denote $t_i = (\Gamma_i, \eta_i)$. As $\langle t_i \wedge t_{i+1} \mid i < \delta \rangle$ is an increasing sequence (because $\bar{t} \in \mathrm{HNind}_{\mathrm{ai}}(M_c)$), $0 < \mathrm{lev}(t_1 \wedge t_2)$. Let $p = \Gamma_{t_1 \wedge t_2}(0) \restriction \emptyset$. Then $p = \Gamma_i(0) \restriction \emptyset$ for all $1 \leq i$ (it may be that $t_1 \wedge t_0 = \emptyset$ and in this case we have no information on t_0). Assume

that $p = \{d(x) = \varepsilon\}$ for some $\varepsilon < \theta$. Then, by Definition 5.17, clauses (3) and (6), $d(\eta_i(\beta)) = \varepsilon$ for all $1 \le i < \delta$ and $\beta < \lg(\eta_i)$. As $u_i = l_{t_i}$ we are done.

(2) Denote $v_i = (\Gamma'_i, \eta'_i)$. By our assumptions on $\bar{t}$, $t_i E^{\mathrm{nb}} v_i$ hence if $\bar{t}$ is increasing then $\bar{v} = \bar{t}$. Assume that it is not increasing. Then $t_i \wedge t_{i+1} < t_i$ so $\lim(t_i \wedge t_{i+1}) = t_i \wedge t_{i+1}$. Let $\alpha_i = \beta_i + 1 = \mathrm{lev}(t_i) = \lg(\eta'_i)$, then β_i is a limit ordinal and $t_i \restriction \beta_i = v_i \restriction \beta_i$. By the same argument as before, for $1 \le i$, $\Gamma'_i(0) \restriction \emptyset = \Gamma_i(0) \restriction \emptyset = p$ and $\Gamma'_i \restriction \beta_i = \Gamma_i \restriction \beta_i$.

Note that for $1 \le i$, l_{t_i} and l_{v_i} are both below $u_{i+1} = l_{t_{i+1}}$ (as $v_i \le t_{i+1}$ and $l_{t_i} = u_i < u_{i+1}$), that they both satisfy p and that they both satisfy the equations in $\Gamma(\beta)$ for each limit $\beta < \beta_i$, so if for instance $l_{t_i} < l_{v_i}$, we will have a contradiction to Definition 5.17, clause (7).

So, in any case (whether or not $\bar{t}$ is increasing), we have $l_{v_i} = l_{t_i} = u_i$.

By choice of $\bar{v}$ and the assumptions on $\bar{t}$, $\bar{v}$ is increasing so d is defined on finite subsets of it.

Assume $1 \le i_0 < \ldots < i_n < \delta$. Then for every $\sigma < \theta$, by the choice of d in Definition 5.17:

- ⊠ $\pi_1\left(d\left(v_{i_0}, \ldots, v_{i_{n-1}}\right)\right) = \sigma$ iff
- ⊠ $d\left(l_{v_{i_0}}, \ldots, l_{v_{i_{n-1}}}, x\right) = \sigma \in \Gamma'_{i_{n-1}}\left(\beta_{i_{n-1}}\right)$ iff
- ⊠ $d\left(l_{v_{i_0}}, \ldots, l_{v_{i_{n-1}}}, x\right) = \sigma \in \Gamma'_{i_n}\left(\beta_{i_{n-1}}\right)$ (because $\Gamma'_{i_n} \restriction \alpha_{i_{n-1}} = \Gamma'_{i_{n-1}} \restriction \alpha_{i_{n-1}}$) iff
- ⊠ $d\left(l_{v_{i_0}}, \ldots, l_{v_{i_{n-1}}}, l_{v_{i_n}}\right) = \sigma$ (this follows from clause (6) of Definition 5.17) iff
- ⊠ $d(u_{i_0}, \ldots, u_{i_n}) = \sigma$ (because $l_{v_i} = u_i$).

□

Corollary 5.26. *If $\bar{s} \in \mathrm{HNind}_{\mathrm{ai}}(M_c)$ then there must be some $n < \omega$ such that $H^{(n)}(\bar{s}) \in \mathrm{HNind}_f(M_c)$ (see Definition 5.11).*

Proof. If not, by Corollary 5.13, for all $n < \omega$, $H^{(n)}(\bar{s}) \in \mathrm{HNind}_{\mathrm{ai}}(M_c)$. Moreover, there exists some $K < \omega$ such that for all $K \le n$, if $\bar{t} = H^{(n)}(\bar{s})$ then $\mathrm{suc}(\lim(t_i \wedge t_{i+1}), t_i) = t_i$. Without loss, $K = 0$ (i.e. this is true also for $\bar{s}$).

Claim. If $\bar{s}$ is such a sequence then $d\left(u_{i_0}, \ldots, u_{i_{n-1}}\right)$ is constant for all $1 \le i_0 < \ldots < i_{n-1} < \delta$ where $u_i = \mathrm{suc}(\lim(s_i \wedge s_{i+1}), s_{i+1})$ for $i < \delta$.

Proof. (of claim) Prove by induction on n using Lemma 5.24. □

But this claim contradicts the fact that for all $k < \omega$, $\mathbf{p}_k$ is hard. □

Lemma 5.27. *If $\bar{s} \in \mathrm{HNind}_{\mathrm{ai}}(M_c)$ and $\bar{t} = H(\bar{s}) \in \mathrm{HNind}_f(M_c)$ then $\neg(v_i E v_j)$ for $i < j < \delta$ where $v_i = \mathrm{suc}(\lim(t_{i+1} \wedge t_i), t_i)$.*

Proof. Let $t = t_0 \wedge t_1$, so $t = t_i \wedge t_j$ for all $i < j < \delta$. Let $u_i = \mathrm{suc}(t, t_i)$. As $t_i \ne t_j$ for $i < j < \delta$, $u_i \ne u_j$. In addition

$$l_{u_i} \le l_{t_i} = \mathrm{suc}(\lim(s_i \wedge s_{i+1}), s_{i+1}) \le s_{i+1} \wedge s_{i+2}$$

and $\langle s_i \wedge s_{i+1} \,|\, i < \delta \rangle$ is increasing so l_{u_i} and l_{u_j} are comparable.

First assume that $\alpha = \operatorname{lev}(t) > 0$. Then $\Gamma_t(0) = \Gamma_{t_i}(0)$ for $i < \delta$. For all $i < j < \delta$, $l_{u_i} \models \Gamma_{u_j}(0) \restriction \emptyset$, l_{u_i} is greater than $\eta_{u_j}(\beta) = \eta_t(\beta)$ for all $\beta < \alpha$ and $l_{u_i} \models \Gamma_{u_j}(\beta) = \Gamma_t(\beta)$ for all limit $\beta < \alpha$. So by Definition 5.17, clause (7), $l_{u_i} = l_{u_j}$, so $\eta_{u_i} = \eta_{u_j}$ for all $i < j < \delta$.

But since $u_i \neq u_j$, it necessarily follows that $\Gamma_{u_i} \neq \Gamma_{u_j}$. If $\alpha = \beta + 1$ for some β, then by choice of $\mathbf{q}$, $\Gamma_{u_i} = \Gamma_{u_i} \restriction \alpha = \Gamma_t$ (because Γ was defined only for limit ordinals). So necessarily α is a limit, and it follows that $\lim(t) = t$ so $v_i = u_i$. Now it is clear that $\Gamma_{v_i}(\alpha) \neq \Gamma_{v_j}(\alpha)$ and by definition of E, $\neg(v_i E v_j)$ for all $i < j < \delta$.

If $\alpha = 0$, then as before $v_i = u_i$ (because $\lim(t) = t$). We cannot use the same argument (because $\Gamma_t(0)$ is not defined), so we take care of each pair $i < j < \delta$ separately. If $\Gamma_{v_i}(0) \restriction \emptyset = \Gamma_{u_i}(0) \restriction \emptyset$ then the argument above will work and $\neg(v_i E v_j)$. If $\Gamma_{v_i}(0) \restriction \emptyset \neq \Gamma_{v_j}(0) \restriction \emptyset$, then $\neg(v_i E v_j)$ follows directly from the definition. □

Finally we have

Corollary 5.28. *If $\bar{s} \in \mathrm{HNind}_{\mathrm{ai}}(M_c)$, then there is some $\bar{v} \in \mathrm{HNind}_f(M_c)$ such that $v_i = \operatorname{suc}(\lim(v_i), v_i)$, $v_i E^{\mathrm{nb}} v_j$ but $\neg(v_i E v_j)$ for $i < j < \delta$.*

Proof. By Corollary 5.26, there is some minimal $n < \omega$ such that $\bar{t} = H^{(n+1)}(\bar{s}) \in \mathrm{HNind}_f(M_c)$. Let $v_i = \operatorname{suc}(\lim(t_{i+1} \wedge t_i), t_i)$ for $i < \delta$. By Lemma 5.27, we have that $v_i E^{\mathrm{nb}} v_j$ but $\neg(v_i E v_j)$ for $i < j < \delta$ (in particular $v_i \neq v_j$). So necessarily $t = t_i \wedge t_j$ is a limit and $v_i = \operatorname{suc}(t, v_i)$. □

Corollary 5.29. *If there is some $\bar{s} \in \operatorname{ind}(M_c)$ and $s_i \in P_0$ for all $i < \delta$, then there is some $\bar{v} \in \operatorname{ind}_f(M_c)$ such that $v_i \in \mathrm{Suc}_{\lim}(M_c)$, $v_i E^{\mathrm{nb}} v_j$ but $\neg(v_i E v_j)$ for $i < j < \delta$.*

Proof. Since $P_0 = \kappa$, any sequence $\bar{s}$ in $\operatorname{ind}(M_c)$ in P_0 must be increasing. So by the last corollary there is some $\bar{v} \in \mathrm{HNind}_f(M_c)$ like there. But then there is some $n < \omega$ such that $\langle v_{ni} \,|\, i < \delta \rangle$ is indiscernible. □

End of the proof of Theorem 4.1.

Assume that M_λ, A_λ are the model and sets guaranteed by the induction hypothesis for $\lambda < \kappa$. We may assume they are disjoint. Let N be a model of $T_S^{\forall,\theta}$ containing M_λ for $\lambda < \kappa$ (N exists by JEP), and let $A = \bigcup \{A_\lambda \,|\, \lambda < \kappa\} \subseteq N$. Let $N_c \models T_\omega^{\forall,\theta}$ be a model such that $N_c \restriction L_S \supseteq M_c$. Let $S' = {}^{\omega>}1$ (finite sequences of zeros). We may think of N_c as a model of $T_{S'}^{\forall,\theta}$. Denote $0_n = \langle 0, \ldots, 0 \rangle$ where $\lg(0_n) = n$.

We use the construction ($\star$) and S' to build a model M of $T_S^{\forall,\theta}$:

- For all $n < \omega$, let $M_{0_n}^1 = N$.
- Define $G_{0_n, 0_n \frown \langle 1 \rangle}^M$ as follows:
  - Recall that $P_{0_n}^M \supseteq P_n^{M_c} = M_{\mathbf{p}_n}$. Assume that $B \subseteq \mathrm{Suc}_{\lim}\left(P_n^{M_c}\right)$ is an E^{nb} class. By definition, $|B / E_{\mathbf{p}_n}| < \kappa$.

- Choose some enumeration of the classes $\{c_i \mid i < |B/E_{\mathbf{p}_n}|\}$, and an enumeration $A_{|B/E_{\mathbf{p}_n}|} \supseteq \{a_i \mid i < |B/E_{\mathbf{p}_n}|\}$ of pairwise distinct elements. Now, $G^M_{0_n,0_n\hat{}\langle 1\rangle}$ maps every class c_i (i.e. every element in c_i) to a_i. Define $G^M_{0_n,0_n\hat{}\langle 1\rangle}$ on other elements in $P_n^{M_c}$ by regressiveness. Outside of $P_n^{M_c}$, define $G^M_{0_n,0_n\hat{}\langle 1\rangle}$ arbitrarily as long as it is regressive.

Let $A = \mathrm{Suc}_{\lim}\left(P_{\langle\rangle}^{M_c}\right)$, i.e. $A = \mathrm{Suc}_{\lim}(\kappa)$. Assume for contradiction that A contains a δ-indiscernible sequence.

By Corollary 5.29, there is $n < \omega$ and an indiscernible sequence $\bar{v}$ in $\mathrm{Suc}_{\lim}\left(P^M_{0_n}\right)$ such that for $i < j < \delta$, $v_i E^{\mathrm{nb}} v_j$ but $\neg(v_i E v_j)$. So $\left\langle G^M_{0_n,0_n\hat{}\langle 1\rangle}(v_i) \mid i < \delta \right\rangle$ is a non-constant indiscernible sequence in $A_{|[v_0/E^{\mathrm{nb}}]/E_{\mathbf{p}_n}|}$ — a contradiction.

6. Further comments

Strongly dependent theories.

Strongly dependent theories are a sub-class of dependent theories. They were introduced and discussed in [She12, She09] as a possible solution to the equation dependent / x = stable / superstable. There, the author also introduced other classes of smaller sub-classes, namely strongly^l dependent for $l = 2, 3, 4$. Groups which are strongly^2 dependent are discussed in [KS11]. The theories of the reals and the p-adics are both strongly dependent, but neither is strongly^2 dependent.

As we said in the introduction, in [She12] it is proved that $\beth_{|T|^+}(\lambda) \to (\lambda^+)_{T,n}$ for strongly dependent T and $n < \omega$.

In [KS12] we show that in $\mathbb{R}$ there is a similar phenomenon to what we have here, but for ω-tuples: there are sets from all cardinalities with no indiscernible sequence of ω-tuples (up to the first strongly inaccessible cardinal). This explains why the theorem mentioned was only proved for $n < \omega$.

The example we described here is not strongly dependent, but it can be modified a bit so that it will be, and then give a similar theorem for strongly dependent theories (or even strongly^2 dependent), but for ω-tuples.

Here are the definitions:

Definition 6.1. A theory T is said to be not *strongly dependent* if there exists a sequence of formulas $\langle \varphi_i(x, y_i) \rangle$ (where x, y_i are tuples of variables), an array $\langle a_{i,j} \mid i, j < \omega \rangle$ ($\lg(a_{i,j}) = \lg(y)$) and tuples $\langle b_\eta \mid \eta : \omega \to \omega \rangle$ ($\lg(b_\eta) = \lg(x)$) such that $\models \varphi_i(b_\eta, a_{i,j}) \Leftrightarrow \eta(i) = j$.

Notation 6.2. We call an array of elements (or tuples) $\langle a_{i,j} \mid i, j < \omega \rangle$ an *indiscernible array* over A if for $i_0 < \omega$, the i_0-row $\langle a_{i_0,j} \mid j < \omega \rangle$ is indiscernible over the rest of the sequence $(\{a_{i,j} \mid i \neq i_0, i, j < \omega\})$ and A, i.e. when the rows are mutually indiscernible.

Definition 6.3. A theory T is said to be not *strongly*^{2} *dependent* if there exists a sequence of formulas $\langle \varphi_i(x, y_i, z_i) \,|\, i < \omega \rangle$, an array $\langle a_{i,j} \,|\, i, j < \omega \rangle$ and $b_k \in \{a_{i,j} \,|\, i < k, j < \omega\}$ such that

- The array $\langle a_{i,j} \,|\, i, j < \omega \rangle$ is an indiscernible array (over $\emptyset$).
- The set $\{\varphi_i(x, a_{i,0}, b_i) \wedge \neg\varphi_i(x, a_{i,1}, b_i) \,|\, i < \omega\}$ is consistent.

Theorem 6.4. *For every θ there is a strongly^2 dependent theory T of size θ such that for all κ and δ, $\kappa \to (\delta)_{T,\omega}$ iff $\kappa \to (\delta)_\theta^{<\omega}$.*

Proof. First note that the easy direction works the same way as before.

For $n < \omega$ let T_n^θ be the theory defined in Definition 3.26 for $S_n = {}^{n\geq}2$. Let T be the theory $\sum_{n<\omega} T_n^\theta$: the language is $\{Q_n \,|\, n < \omega\} \cup \left\{R^n \,\middle|\, R \in L_{S_n}^\theta\right\}$ where Q_n are unary predicates, and the theory says that they are mutually disjoint and that each Q_n is a model of T_n^θ. It is easy to see that this theory is complete and has quantifier elimination. Denote $S = {}^{\omega\geq}2$ as before. If M is a model of T_S^θ, then M naturally induces a model N of T (where $Q_n^N = (M \times \{n\}) \upharpoonright L_{S_n}^\theta$). For all $a \in M$, let $f_a \in \prod_{n<\omega} Q_n^N$ be defined by $f_a(n) = (a, n)$ for $n < \omega$. Now, if $A \subseteq P_{\langle\rangle}^M$ is any set with no δ-indiscernible sequence then the set $\{f_a \,|\, a \in A\}$ is a sequence of ω-tuples with no indiscernible sequence of length δ.

By Remark 3.24, it follows that each T_n is strongly^2 dependent, and so also T. $\square$

Comments about the proof.

We would like to explain some of the choices made through the proof.

(1) Why did we use discrete trees and not dense ones (as in [KS12])?

This is because indiscernible sequences could be in a diagonal comb situation (i.e. in $\mathrm{ind}_{\mathrm{ai}}$), and we needed the ability to make an element be a successor to the meet (otherwise we have no control on its coloring). Trial and error has shown that adding the function "successor to the meet" instead of just successor causes the lost of amalgamation, so we needed the successor function. The predecessor function is not necessary (in existentially closed models, if $x > \lim_\eta(x)$, x has a predecessor), but there is no price to adding it, and it simplify the theory a bit.

(2) Why did we need the regressiveness in the definition of $T_S^\forall$?

Because otherwise there is no quantifier elimination: for $a <_{\eta_1} b$ we can ask whether

$$\exists x \left(a <_{\eta_1} x <_{\eta_1} b \wedge G_{\eta_1,\eta_2}(x) \neq G_{\eta_1,\eta_2}(b)\right).$$

This formula is not equivalent to a quantifier free formula without regressiveness, because for every m we can find a, b and a', b' such that $ab \equiv_m a'b'$ but one of the pairs satisfies the formula and the other does not.

(3) Why in the definition of $\mathbf{q}$ we demanded that the image of η is in $\mathrm{Suc}_{\lim}$ and that Γ is relevant only in limit levels?

Had we given Γ the freedom to give values in every ordinal, then the "fan" (i.e. the sequence in ind_f) which we got in the end (in Lemma 5.27) might not have been in a successor to a limit level, so we would have no freedom in applying G on it. As Γ is relevant only for limit levels, the coloring was defined only on sequence in $\mathrm{Suc}_{\mathrm{lim}}$, so we needed η to give elements from there.

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